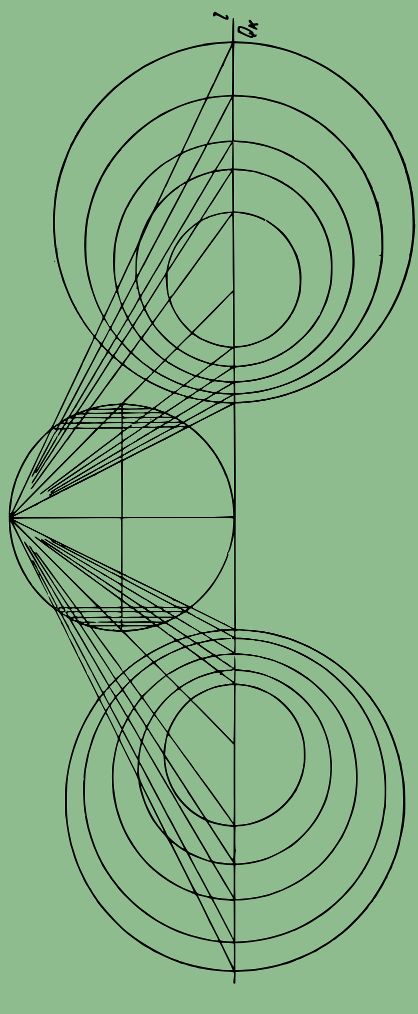
P.S. MODENOV

PROBLEMS IN GEOMETRY



ИЗДАТЕЛЬСТВО МИР

П. С. МОДЕНОВ

ЗАДАЧИ ПО ГЕОМЕТРИИ

МОСКВА «НАУКА»

P.S. MODENOV

PROBLEMS IN GEOMETRY

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CONTENTS

PREFACE

This text offers certain general methods of solving problems in elementary geometry and is designed for teachers of mathematics in secondary schools and also for senior students.

The present text includes material that goes beyond the scope of mathematics curricula for secondary schools (the use of complex numbers in plane geometry, inversion, pencils of circles and others).

The book consists of five chapters. The first four chapters deal with the application of vector algebra, analytic geometry, complex numbers and the inversion transformation to geometric problems. Chapter V contains a list of the basic definitions and formulas used in the first four chapters. Before starting a new chapter, the reader is advised to refresh his memory with the appropriate material of Chapter V. Some of the derivations of formulas given in Chapter V are familiar to senior students of secondary school. More detailed theoretical material can be found in the bibliography at the end of the book.

I wish here to remark on a supplement to vector algebra that was brought to my attention in 1930 by Professor Ya. S. Dubnov, my teacher at Moscow State University. It is that vector algebra in the plane has not been developed to the point that vector algebra in space has, and in order to remedy this situation in an oriented plane it is necessary to introduce the rotation of a vector through an angle of $+\pi/2$ (designated $[\mathbf{a}]$) and also a pseudoscalar (or cross) product $\mathbf{a} \times \mathbf{b}$ [or $(\mathbf{a}, \mathbf{b})$] of a vector $\mathbf{a}$ by a vector $\mathbf{b}$. Note that the linear vector function $A\mathbf{x}$ of a vector argument $\mathbf{x}$ possessing the property that $A\mathbf{x} \perp \mathbf{x}$ for any vector $\mathbf{x}$ has the form $A\mathbf{x} = \lambda[\mathbf{x}]$ (λ is an arbitrary number) in the plane, and $A\mathbf{x} = [\mathbf{a}, \mathbf{x}]$ ($\mathbf{a}$ is an arbitrary vector) in space. The cross product of vectors in the plane and in space may be defined as a polylinear scalar function (of two vectors in the plane and of three vectors in space) which is antisymmetric with respect to any pair of vectors – in the plane we have

$$A(\mathbf{x}, \mathbf{y}) = -A(\mathbf{y}, \mathbf{x});$$

in space we have

$$A(\mathbf{x}, \mathbf{y}, \mathbf{z}) = -A(\mathbf{y}, \mathbf{x}, \mathbf{z}), \quad A(\mathbf{x}, \mathbf{y}, \mathbf{z}) = -A(\mathbf{z}, \mathbf{y}, \mathbf{x}),$$
$$A(\mathbf{x}, \mathbf{y}, \mathbf{z}) = -A(\mathbf{x}, \mathbf{z}, \mathbf{y})$$

— and is normed (that is, it becomes $+1$ for some base).

This product may be defined as the result of two operations (in the plane and in space)

$$(\mathbf{a}, \mathbf{b}) = [\mathbf{a}] \cdot \mathbf{b}, \ (\mathbf{a}, \mathbf{b}, \mathbf{c}) = [\mathbf{a}, \mathbf{b}] \cdot \mathbf{c}.$$

Although a free vector in geometry constitutes the *class* of all equivalent directed line segments, I will permit myself, in this book (in accordance with a very solid tradition) to identify a vector and a directed line segment as equal (as, for example, in arithmetic, where one regards as equal the fractions p/q and np/nq, where p, q, n are natural numbers). For this reason, in this text, two directed line segments that are collinear, have the same length, and are in the same direction will be termed equivalent or equal.

The idea of using complex numbers in plane geometry came to me in connection with some very interesting lectures on the theory of analytic functions delivered at Moscow University by Professor A. I. Markushevich, and also with a book on that subject by Markushevich. Also, since the 1940 s, papers have appeared regularly in mathematical journals in many countries illustrating how the use of complex numbers in plane geometry makes for rather simple solutions to complicated problems by relating the solutions to basic geometric transformations that are normally studied in secondary school (motion, the similarity transformation, circular transformations, including inversion).

A book by R. Deaux [2] appeared in France devoted specially to the problems taken up in Chapter III of this book. Since this methodology is not all represented in Soviet textbooks, I have given detailed explanations and calculations of the procedures. In this text I have made use of the work of R. Deaux, R. Blanchard, Gourmagschieg, V. Jebeau and others.

I believe that the contents of Chapter III is added proof of how much elementary mathematics loses if complex numbers are not brought into the picture. A consideration of the most elementary functions of a complex variable,

$$z' = \frac{az + b}{az + d}(ad - bc \neq 0), \qquad z' = \frac{a\bar{z} + c}{cz + d}(ad - bc \neq 0),$$

$$z' = az + b(a \neq 0), \qquad z' = a\bar{z} + b(a \neq 0),$$

embraces the isometric transformations of the first and second kind ($z' = az + b$, $z' = \bar{a}\bar{z} + b$, where $|a| = 1$), similarity transformations of

the first and second kind ($z' = az + b$, $z' = a\bar{z} + b, a \neq 0$) and circular transformations (the case of a linear fractional function; in particular, the inversion $z' = a/z$).

Chapter IV gives a survery of the properties of inversion of a plane and space and various applications (inversors, the geometry of Mascheroni, and the mapping of regions under inversion). In particular, detailed consideration is given to various stereographic projections of a sphere onto a plane and the construction of conformal maps of a spectrum of meridians and parallels of the sphere.

The final chapter, Chapter V, contains a list of basic definitions, formulas and the bibliography. The bibliography contains books in which the reader will find proofs of the formulas used in this text; they include textbooks on vector algebra, analytic geometry, the theory of geometric transformations, and the theory of functions of a complex variable.

The general methods for solving geometric problems described in this text are closely interrelated: it will be recalled that vector algebra is closely related with analytic geometry. The basic formulas used in Chapter III are derived on the basis of facts taken from analytic geometry; the linear fractional function of a complex variable contains within it the inversion transformation; the inversion transformation can be reliably studied by the methods of analytic geometry, and so forth.

I would like to point out that the drawing on the cover of the book (it is the same as that in Fig. 114) is a copy of a *photograph* of a model that I constructed to illustrate the stereographic projection of a sphere onto a plane under which the parallels and meridians pass into a hyperbolic pencil of circles and an associated elliptical pencil of circles. Figures 107 and 108 were done in the same manner.

During the writing of this text I received valuable advice from Professor V. A. Ilin and Corresponding Member of the USSR Academy of Sciences S. V. Yablonsky, to whom I express my deep gratitude. Very profound and valuable advice was obtained from the reviewer of the Nauka Publishing House; practically all his suggestions were incorporated in the final version of the manuscript.

It goes without saying that the general methods of solving elementary-geometry problems given in this text do not exhaust the range of such methods. For instance, mention may be made of a very powerful analytic method for applying trilinear coordinates in the plane, and tetrahedral coordinates in space (the trilinear coordinates of a point on a projective-Euclidean plane are the projective coordinates of proper points of such a plane, provided that all four fundamental points of the projective system

of coordinates are also proper points; the same goes for space as well). The limited scope of this book did not allow for the inclusion of that method. And there are of course other general methods, which, unfortunately, have not been discussed in our textbooks or teaching literature (for example, synthetic methods of solving problems with the use of isometric, similarity, affine and projective transformations). However, I am sure that this situation will be remedied in time.

P. S. Modenov

The present edition was prepared after the author died. The material of the book has been re-examined and brought into accord with generally accepted terminology and notation. A small number of inaccuracies in the Russian edition have been corrected and the bibliography has been expanded.

Chapter I

VECTOR ALGEBRA

Sec. 1. Vectors in the plane (solved problems)

Problem 1. Given the angles A, B, C of $\triangle ABC$. Find $\angle\varphi = \angle BAM$, where M is the midpoint of BC.

Solution.

$$\overrightarrow{AM} \uparrow\uparrow (\overrightarrow{AB} + \overrightarrow{AC})$$

and so

$$\cos\varphi = \frac{\overrightarrow{AB}(\overrightarrow{AB} + \overrightarrow{AC})}{|\overrightarrow{AB}|\,|\overrightarrow{AB} + \overrightarrow{AC}|} = \frac{AB^2 + \overrightarrow{AB}\cdot\overrightarrow{AC}}{c\sqrt{c^2 + b^2 + 2bc\cos A}}$$

$$= \frac{c + b\cos A}{\sqrt{b^2 + c^2 + 2bc\cos A}},$$

and since $b : c = \sin B : \sin C$, it follows that

$$\cos\varphi = \frac{\sin C + \sin B \sin A}{\sqrt{\sin^2 B + \sin^2 C + 2\sin B \sin C \cos A}}.$$

Problem 2. Given the angles A, B, C of $\triangle ABC$. Let M be the midpoint of segment AB, and let D be the foot of the bisector of $\angle C$. Find the ratio $(CDM) : (ABC)$ and also $\varphi = \angle DCM$.

Solution.

$$\overrightarrow{CD} = \frac{a\mathbf{b} + b\mathbf{a}}{a + b}, \quad \overrightarrow{CM} = \frac{\mathbf{a} + \mathbf{b}}{2},$$

where $\mathbf{a} = \overrightarrow{CB}$, $\mathbf{b} = \overrightarrow{CA}$. Consequently

$$(CDM) = \frac{1}{2}(\overrightarrow{CD}, \overrightarrow{CM}) = \frac{(a\mathbf{b} + b\mathbf{a},\ \mathbf{a} + \mathbf{b})}{4(a + b)}$$

$$= \frac{(b - a)(\mathbf{a}, \mathbf{b})}{4(a + b)} = \frac{(a - b)(ABC)}{2(a + b)}$$

whence

$$\frac{(CDN)}{(ABC)} = \frac{a - b}{2(a + b)},$$

which can also be written as

$$\frac{(CDM)}{(ABC)} = \frac{\sin A - \sin B}{2(\sin A + \sin B)}.$$

Furthermore, since $\overrightarrow{CD} \uparrow\uparrow (a\mathbf{b} + b\mathbf{a})$, $\overrightarrow{CM} \uparrow\uparrow (\mathbf{a} + \mathbf{b})$, it follows that

$$\cos\varphi = \frac{(a\mathbf{b} + b\mathbf{a})(\mathbf{a} + \mathbf{b})}{|a\mathbf{b} + b\mathbf{a}|\,|\mathbf{a} + \mathbf{b}|}$$

$$= \frac{ab^2 + ba^2 + ab(a+b)\cos C}{\sqrt{2a^2b^2 + 2a^2b^2\cos C}\,\sqrt{a^2 + b^2 + 2ab\cos C}}$$

$$= \frac{(a+b)\cos(C/2)}{\sqrt{a^2 + b^2 + 2ab\cos C}} = \frac{(\sin A + \sin B)\cos(C/2)}{\sqrt{\sin^2 A + \sin^2 B + 2\sin A\sin B\cos C}},$$

$$\sin\varphi = \frac{(a\mathbf{b} + b\mathbf{a},\ \mathbf{a} + \mathbf{b})}{|a\mathbf{b} + b\mathbf{a}|\,|\mathbf{a} + \mathbf{b}|} = \frac{(b-a)(\mathbf{a},\mathbf{b})}{ab\sqrt{2(1+\cos C)}\,\sqrt{a^2 + b^2 + 2ab\cos C}}$$

$$= \frac{(\sin B - \sin A)\sin(C/2)}{\sqrt{\sin^2 A + \sin^2 B + 2\sin A\sin B\cos C}}.$$

Problem 3. Given the interior angles A, B, C of $\triangle ABC$; M is the midpoint of segment BC, N is the foot of the altitude dropped from point C to side AB, and O is the point of intersection of the straight lines AM and CN. Find $\cos\varphi$, where $\varphi = \angle AOC$.

Solution. Orientate the plane with the base $\mathbf{a}$, $\mathbf{b}$, where $\mathbf{a} = \overrightarrow{CB}$, $\mathbf{b} = \overrightarrow{CA}$. Then $\overrightarrow{CN} \uparrow\uparrow [\mathbf{a} - \mathbf{b}]$. Indeed, the vector $[\mathbf{a} - \mathbf{b}] = [\overrightarrow{BA}]$ is perpendicular to the straight line AB and forms acute angles with the vectors $\mathbf{a}$ and $\mathbf{b}$ since

$$[\mathbf{a} - \mathbf{b}]\,\mathbf{a} = -[\mathbf{b}]\,\mathbf{a} = (\mathbf{a},\ \mathbf{b}) > 0,$$

$$[\mathbf{a} - \mathbf{b}]\,\mathbf{b} = [\mathbf{a}]\,\mathbf{b} = (\mathbf{a},\ \mathbf{b}) > 0.$$

Furthermore, $\overrightarrow{AM} \uparrow\uparrow \frac{\mathbf{a}}{2} - \mathbf{b} \uparrow\uparrow \mathbf{a} - 2\mathbf{b}$.

The desired $\angle\varphi$ is the angle between the vectors $\overrightarrow{AM}$ and $\overrightarrow{CN}$; consequently,

$$\cos\varphi = \frac{[\mathbf{a} - \mathbf{b}](\mathbf{a} - 2\mathbf{b})}{|[\mathbf{a} - \mathbf{b}]|\,|\mathbf{a} - 2\mathbf{b}|} = \frac{-2[\mathbf{a}]\mathbf{b} - [\mathbf{b}]\mathbf{a}}{|\mathbf{a} - \mathbf{b}|\,|\mathbf{a} - 2\mathbf{b}|} =$$

$$= \frac{-2(\mathbf{a},\ \mathbf{b}) + (\mathbf{a},\ \mathbf{b})}{|\mathbf{a} - \mathbf{b}|\,|\mathbf{a} - 2\mathbf{b}|} = -\frac{(\mathbf{a},\ \mathbf{b})}{|\mathbf{a} - \mathbf{b}|\,|\mathbf{a} - 2\mathbf{b}|}$$

$$= -\frac{ab \sin C}{\sqrt{a^2 + b^2 - 2ab\cos C}\,\sqrt{a^2 + 4b^2 - 4ab\cos C}}$$

$$= -\frac{\sin A \sin B \sin C}{\sqrt{\sin^2 A + \sin^2 B - 2\sin A \sin B \cos C}\sqrt{\sin^2 A + 4\sin^2 B - 4\sin A \sin B \cos C}}.$$

Problem 4. Given vectors $\mathbf{a} = \overrightarrow{CB}$, $\mathbf{b} = \overrightarrow{CA}$. Find the vector $\mathbf{x} = \overrightarrow{CO}$, where O is the centre of a circle circumscribed about $\triangle ABC$.

Solution. From the relations

$$\mathbf{x}^2 = (\mathbf{a} - \mathbf{x})^2 = (\mathbf{b} - \mathbf{x})^2$$

we find

$$\mathbf{xa} = a^2/2, \quad \mathbf{xb} = b^2/2$$

and, hence, by the Gibbs formula

$$\mathbf{x} = \mathbf{xa}\cdot\mathbf{a}^* + \mathbf{xb}\cdot\mathbf{b}^* = \frac{a^2}{2}\,\mathbf{a}^* + \frac{b^2}{2}\,\mathbf{b}^*$$

where $\mathbf{a}^*$, $\mathbf{b}^*$ is the reciprocal (or dual) basis of the basis $\mathbf{a}$, $\mathbf{b}$:

$$\mathbf{a}^* = \frac{[\mathbf{b}]}{(\mathbf{b}, \mathbf{a})}, \quad \mathbf{b}^* = \frac{[\mathbf{a}]}{(\mathbf{a}, \mathbf{b})}.$$

Thus,

$$\mathbf{x} = \frac{b^2[\mathbf{a}] - a^2[\mathbf{b}]}{(\mathbf{a}, \mathbf{b})}.$$

Remark. If we take $\mathbf{x}$ in the form $\mathbf{x} = \lambda\mathbf{a} + \mu\mathbf{b}$, then from the relations

$$\mathbf{xa} = a^2/2, \quad \mathbf{xb} = b^2/2$$

we obtain

$$\lambda a^2 + \mu\cdot\mathbf{ab} = a^2/2,$$
$$\lambda\cdot\mathbf{ab} + \mu b^2 = b^2/2,$$

whence

$$\lambda = \frac{a^2b^2 - b^2\cdot\mathbf{ab}}{2(a^2b^2 - (\mathbf{ab})^2)}, \quad \mu = \frac{a^2b^2 - a^2\cdot\mathbf{ab}}{2(a^2b^2 - (\mathbf{ab})^2)}$$

and so

$$\mathbf{x} = \frac{1}{2}\,\frac{a^2b^2 - b^2\cdot\mathbf{ab}}{a^2b^2 - (\mathbf{ab})^2}\,\mathbf{a} + \frac{1}{2}\,\frac{a^2b^2 - a^2\cdot\mathbf{ab}}{a^2b^2 - (\mathbf{ab})^2}\,\mathbf{b}.$$

Problem 5. Given the cross products $(\mathbf{a}, \mathbf{x}) = p$, $(\mathbf{b}, \mathbf{x}) = q$ of vector $\mathbf{x}$ into the noncollinear vectors $\mathbf{a}$ and $\mathbf{b}$. Express the vector $\mathbf{x}$ in terms of the vectors $\mathbf{a}$, $\mathbf{b}$ and the numbers p, q.

Solution. Let

$$\mathbf{a}^* = \frac{[\mathbf{b}]}{(\mathbf{b},\ \mathbf{a})}, \quad \mathbf{b}^* = \frac{[\mathbf{a}]}{(\mathbf{a},\ \mathbf{b})}$$

be the reciprocal basis of $\mathbf{a}$, $\mathbf{b}$. Then

$$\mathbf{x} = (\mathbf{x}\mathbf{a}^*)\mathbf{a} + (\mathbf{x}\mathbf{b}^*)\mathbf{b} = \left(\frac{[\mathbf{b}]}{(\mathbf{b},\ \mathbf{a})},\ \mathbf{x}\right)\mathbf{a} + \left(\frac{[\mathbf{a}]}{(\mathbf{a},\ \mathbf{b})},\ \mathbf{x}\right)\mathbf{b}$$

$$= \frac{(\mathbf{b},\ \mathbf{x})}{(\mathbf{b},\ \mathbf{a})}\mathbf{a} + \frac{(\mathbf{a},\ \mathbf{x})}{(\mathbf{a},\ \mathbf{b})}\mathbf{b} = \frac{q}{(\mathbf{b},\ \mathbf{a})}\mathbf{a} + \frac{p}{(\mathbf{a},\ \mathbf{b})}\mathbf{b} = \frac{p\mathbf{b} - q\mathbf{a}}{(\mathbf{a},\ \mathbf{b})}.$$

Problem 6. Two forces $\mathbf{F}_1 = \{2, 3\}$ and $\mathbf{F}_2 = \{4, 1\}$ are specified relative to a general Cartesian system of coordinates. Their points of application are, respectively, $A = (1,1)$ and $B = (2, 4)$. Find the coordinates of the resultant and the equation of the straight line l containing it.

Solution. The coordinates of the resultant $\mathbf{F}$ are 6 and 4. Now let $M(x, y)$ be an arbitrary point of l. Then the moment of the resultant about point M is equal to zero. This moment is equal to the sum of the moments $(\overrightarrow{MA}, \mathbf{F}_1)$ and $(\overrightarrow{MB}, \mathbf{F}_2)$ of component forces (the cross product of vectors is distributive).

Since $\overrightarrow{MA} = \{1 - x,\ 1 - y\}$, $\overrightarrow{MB} = \{2 - x,\ 4 - y\}$, it follows that

$$(\overrightarrow{MA}, \mathbf{F}_1) = \sqrt{g}\begin{vmatrix} 1 - x & 2 \\ 1 - y & 3 \end{vmatrix}, \quad (\overrightarrow{MB}, \mathbf{F}_2) = \sqrt{g}\begin{vmatrix} 2 - x & 4 \\ 4 - y & 1 \end{vmatrix}$$

and, hence, the equation of the straight line l is

$$\begin{vmatrix} 1 - x & 2 \\ 1 - y & 3 \end{vmatrix} + \begin{vmatrix} 2 - x & 4 \\ 4 - y & 1 \end{vmatrix} = 0$$

or

$$4x - 6y + 13 = 0.$$

Sec. 2. Vectors in space (solved problems)

Problem 1. The plane angles of a trihedral angle $OABC$ are $a = \angle BOC$, $b = \angle COA$, $c = \angle AOB$. The interior dihedral angles of the given trihedral angle are:

$$A = B(OA)C, \quad B = C(OB)A, \quad C = A(OC)B^*$$

A trihedral angle $OA^*B^*C^*$ that is the *reciprocal* of the trihedral angle $OABC$ is a trihedral angle constructed in the following manner: ray OA^*

* The symbol $B(OA)C$ is used to denote a dihedral angle with edge OA, in the half-planes of which are points B and C.

is perpendicular to the rays OB and OC and forms an acute angle with ray OA. The rays OB^* and OC^* are constructed in similar fashion.

Let a^*, b^*, c^* be the plane angles of the trihedral angle $OA^*B^*C^*$ and let A^*, B^*, C^*, be its interior dihedral angles.

1°. Knowing, a, b, c, find $\cos A$, $\cos B$, $\cos C$.

2°. Prove that $a^* = \pi - A$, $b^* = \pi - B$, $c^* = \pi - C$.

3°. Prove that $A^* = \pi - a$, $B^* = \pi - b$, $C^* = \pi - c$.

4°. Knowing A, B, C, find $\cos a$, $\cos b$, $\cos c$.

5°. Prove that

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{\Delta}{\sin a \sin b \sin c}$$

where

$$\Delta = \left(\begin{vmatrix} 1 & \cos b & \cos c \\ \cos b & 1 & \cos a \\ \cos c & \cos a & 1 \end{vmatrix}\right)^{1/2}$$

$$= \sqrt{1 + 2\cos a \cos b \cos c - \cos^2 a - \cos^2 b - \cos^2 c}$$

(this relation is called the *theorem of sines for a trihedral angle* $OABC$)[1],

6°. Prove that the sine theorem for the trihedral angle $OABC$ (see item 5°) may be written in the form

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{\Delta^*}{\Delta}$$

where

$$\Delta^* = \sqrt{1 + 2\cos a^* \cos b^* \cos c^* - \cos^2 a^* - \cos^2 b^* - \cos^2 c^*}$$

$$= \sqrt{1 - 2\cos A \cos B \cos C - \cos^2 A - \cos^2 B - \cos^2 C}.$$

Solution. 1°. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be the direction vectors of the rays $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ ($\mathbf{e}_1 \uparrow\uparrow \overrightarrow{OA}$, $\mathbf{e}_2 \uparrow\uparrow \overrightarrow{OB}$, $\mathbf{e}_3 \uparrow\uparrow \overrightarrow{OC}$). Then the vectors $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ of the reciprocal basis of the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the direction vectors of the rays $\overrightarrow{OA^*}$, $\overrightarrow{OB^*}$, $\overrightarrow{OC^*}$. We assume the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to be unit vectors and lay them off from point O; then their endpoints E_1, E_2, E_3 will lie on the ray $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ respectively. Through O draw a plane π perpendicular to the ray $\overrightarrow{OC}$. Let E_1^0 and E_2^0 be orthogonal projections of the points E_1 and E_2 on the plane π. Then $C = A(OC)B = \angle E_1^0 O E_2^0$. Consider the vectors

$$\mathbf{e}_1^0 = \overrightarrow{OE_1^0}, \quad \mathbf{e}_2^0 = \overrightarrow{OE_2^0}.$$

[1] Or for a *spherical triangle* cut out of a sphere with centre O by a trihedral angle.

We have

$$\mathbf{e}_1^0 = \mathbf{e}_1 + \lambda\mathbf{e}_3, \quad \mathbf{e}_2^0 = \mathbf{e}_2 + \mu\mathbf{e}_3.$$

Forming the scalar product of both sides of each of these relations by the vector $\mathbf{e}_3$, we obtain

$$0 = \cos b + \lambda, \quad 0 = \cos a + \mu,$$

so that

$$\mathbf{e}_1^0 = \mathbf{e}_1 - \mathbf{e}_3 \cos b, \quad \mathbf{e}_2^0 = \mathbf{e}_2 - \mathbf{e}_3 \cos a$$

and consequently

$$\cos C = \frac{\mathbf{e}_1^0 \cdot \mathbf{e}_2^0}{|\mathbf{e}_1^0|\,|\mathbf{e}_2^0|} = \frac{(\mathbf{e}_1 - \mathbf{e}_3 \cos b)(\mathbf{e}_2 - \mathbf{e}_3 \cos a)}{\sqrt{(\mathbf{e}_1 - \mathbf{e}_3 \cos b)^2}\sqrt{(\mathbf{e}_2 - \mathbf{e}_3 \cos a)^2}}$$

$$= \frac{\cos c - \cos b \cos a - \cos a \cos b + \cos b \cos a}{\sqrt{1 - 2\cos^2 b + \cos^2 b}\ \sqrt{1 - 2\cos^2 a + \cos^2 a}}$$

$$= \frac{\cos c - \cos a \cos b}{\sin a \sin b}.$$

In similar fashion we calculate $\cos A$ and $\cos B$. Thus,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

$$\cos B = \frac{\cos b - \cos c \cos a}{\sin c \sin a}$$

$$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}$$

2°. The formulas obtained in item 1° can be rewritten thus:

$$\cos A = \frac{[\mathbf{e}_1\mathbf{e}_2]\cdot[\mathbf{e}_1\mathbf{e}_3]}{|[\mathbf{e}_1\mathbf{e}_2]|\,|[\mathbf{e}_1\mathbf{e}_3]|},$$

$$\cos B = \frac{[\mathbf{e}_2\mathbf{e}_1]\cdot[\mathbf{e}_2\mathbf{e}_3]}{|[\mathbf{e}_2\mathbf{e}_1]|\,|[\mathbf{e}_2\mathbf{e}_3]|},$$

$$\cos C = \frac{[\mathbf{e}_3\mathbf{e}_1]\cdot[\mathbf{e}_3\mathbf{e}_2]}{|[\mathbf{e}_3\mathbf{e}_1]|\,|[\mathbf{e}_3\mathbf{e}_2]|}.$$

Note that in this notation the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ need not necessarily be regarded as unit vectors because when $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are replaced respectively by $\lambda\mathbf{e}_1, \mu\mathbf{e}_2, \nu\mathbf{e}_3$, where $\lambda > 0$, $\mu > 0$, $\nu > 0$, the right-hand members of these relations remain unchanged. Thus, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ may be regarded as any direction vectors of the rays $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$.

These last relations can be rewritten as

$$\cos A = -\frac{\mathbf{e}^2 \cdot \mathbf{e}^3}{|\mathbf{e}^2|\,|\mathbf{e}^3|}, \quad \cos B = -\frac{\mathbf{e}^3 \cdot \mathbf{e}^1}{|\mathbf{e}^3|\,|\mathbf{e}^1|}, \quad \cos C = -\frac{\mathbf{e}^1 \cdot \mathbf{e}^2}{|\mathbf{e}^1|\,|\mathbf{e}^2|}. \tag{a}$$

Indeed,

$$-\frac{\mathbf{e}^2 \cdot \mathbf{e}^3}{|\mathbf{e}^2|\,|\mathbf{e}^3|} = -\frac{\dfrac{[\mathbf{e}_3\mathbf{e}_1]}{\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3} \cdot \dfrac{[\mathbf{e}_1\mathbf{e}_2]}{\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3}}{\left|\dfrac{[\mathbf{e}_3\mathbf{e}_1]}{\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3}\right| \left|\dfrac{[\mathbf{e}_1\mathbf{e}_2]}{\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3}\right|} = \frac{[\mathbf{e}_1\mathbf{e}_3]\cdot[\mathbf{e}_1\mathbf{e}_2]}{|[\mathbf{e}_1\mathbf{e}_3]|\,|[\mathbf{e}_1\mathbf{e}_2]|} = \cos A$$

and similarly for the other two formulas.

Note also the formulas for $\cos a^*$, $\cos b^*$, $\cos c^*$:

$$\cos a^* = \frac{(\mathbf{e}^2, \mathbf{e}^3)}{|\mathbf{e}^2|\,|\mathbf{e}^3|}, \quad \cos b^* = \frac{(\mathbf{e}^3, \mathbf{e}^1)}{|\mathbf{e}^3|\,|\mathbf{e}^1|}, \quad \cos c^* = \frac{(\mathbf{e}^1, \mathbf{e}^2)}{|\mathbf{e}^1|\,|\mathbf{e}^2|}. \tag{b}$$

From formulas (a) and (b) we conclude that

$$\cos A = -\cos a^*, \quad \cos B = -\cos b^*, \quad \cos C = -\cos c^*$$

and since all the angles A, B, C, a^*, b^*, c^* lie in the interval $(0, \pi)$, it follows that

$$a^* = \pi - A, \quad b^* = \pi - B, \quad c^* = \pi - C.$$

These relations can also be derived geometrically.

3°. Writing down the formulas obtained above,

$$\cos A = -\frac{\mathbf{e}^2 \cdot \mathbf{e}^3}{|\mathbf{e}^2|\,|\mathbf{e}^3|}, \quad \cos B = -\frac{\mathbf{e}^3 \cdot \mathbf{e}^1}{|\mathbf{e}^3|\,|\mathbf{e}^1|}, \quad \cos C = -\frac{\mathbf{e}^1 \cdot \mathbf{e}^2}{|\mathbf{e}^1|\,|\mathbf{e}^2|}$$

we have for the trihedral angle $OA^*B^*C^*$

$$\cos A^* = -\frac{\mathbf{e}_2 \cdot \mathbf{e}_3}{|\mathbf{e}_2|\,|\mathbf{e}_3|}, \quad \cos B^* = -\frac{\mathbf{e}_3 \cdot \mathbf{e}_1}{|\mathbf{e}_3|\,|\mathbf{e}_1|}, \quad \cos C^* = -\frac{\mathbf{e}_1 \cdot \mathbf{e}_2}{|\mathbf{e}_1|\,|\mathbf{e}_2|},$$

and since $|\mathbf{e}_1| = |\mathbf{e}_2| = |\mathbf{e}_3| = 1$, it follows that

$$\cos A^* = -\cos a, \quad \cos B^* = -\cos b, \quad \cos C^* = -\cos c,$$

whence

$$A^* = \pi - a, \quad B^* = \pi - b, \quad C^* = \pi - c.$$

4°. Applying the formulas of item 1° to the trihedral angle $OA^*B^*C^*$, we obtain

$$\cos A^* = \frac{\cos a^* - \cos b^* \cos c^*}{\sin b^* \sin c^*}$$

or

$$\cos(\pi - a) = \frac{\cos(\pi - A) - \cos(\pi - B)\cos(\pi - C)}{\sin(\pi - B)\sin(\pi - C)}$$

or

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}$$

The formulas for $\cos b$ and $\cos c$ are derived in similar fashion. Thus

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

$$\cos b = \frac{\cos B + \cos C \cos A}{\sin C \sin A},$$

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B}.$$

5°. From the formula

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

we get

$$\sin A = \sqrt{1 - \cos^2 A}$$

$$= \frac{\sqrt{1 + 2\cos a \cos b \cos c - \cos^2 a - \cos^2 b - \cos^2 c}}{\sin b \sin c} = \frac{\Delta}{\sin b \sin c}$$

whence

$$\frac{\sin A}{\sin a} = \frac{\Delta}{\sin a \sin b \sin c}.$$

The ratios $\frac{\sin B}{\sin b}$ and $\frac{\sin C}{\sin c}$ have the same value. Thus

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{\Delta}{\sin a \sin b \sin c}.$$

6°. Let us write down the sine theorem for the trihedral angle $OA^*B^*C^*$, which is the reciprocal of the trihedral angle $OABC$:

$$\frac{\sin A^*}{\sin a^*} = \frac{\sin B^*}{\sin b^*} = \frac{\sin C^*}{\sin c^*} = \frac{\Delta^*}{\sin a^* \sin b^* \sin c^*}$$

or (see items 2° and 3°)

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = \frac{\Delta^*}{\sin A \sin B \sin C}$$

where

$$\Delta^* = \left(\begin{vmatrix} 1 & \cos b^* & \cos c^* \\ \cos b^* & 1 & \cos a^* \\ \cos c^* & \cos a^* & 1 \end{vmatrix} \right)^{1/2}$$

$$= \sqrt{1 + 2\cos a^* \cos b^* \cos c^* - \cos^2 a^* - \cos^2 b^* - \cos^2 c^*}$$

$$= \sqrt{1 - 2\cos A \cos B \cos C - \cos^2 A - \cos^2 B - \cos^2 C}.$$

From the equations

$$\frac{\sin A}{\sin a} = \frac{\Delta}{\sin a \sin b \sin c},$$

$$\frac{\sin a}{\sin A} = \frac{\Delta^*}{\sin A \sin B \sin C},$$

we obtain the following by thermwise division:

$$\frac{\sin^2 A}{\sin^2 a} = \frac{\Delta}{\Delta^*} \frac{\sin A \sin B \sin C}{\sin a \sin b \sin c} = \frac{\Delta}{\Delta^*} \frac{\sin^3 A}{\sin^3 a}$$

whence

$$\frac{\sin A}{\sin a} = \frac{\Delta^*}{\Delta}.$$

Similarly,

$$\frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{\Delta^*}{\Delta},$$

To summarize,

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{\Delta^*}{\Delta}.$$

Problem 2. Given, in a parallelepiped, the lengths a, b, c of its edges OA, OB, OC and the plane angles between them:

$$\angle BOC = \alpha, \quad \angle COA = \beta, \quad \angle AOB = \gamma.$$

1°. Find the length d of the diagonal OD of the parallelepiped.

2°. Find cosines of the angles φ_1, φ_2, φ_3 formed by the diagonal $\overrightarrow{OD}$ and the edges $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$.

3°. If the coordinates u, v, w of the projections of $\overrightarrow{OD}$ on the axes $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are also given, then prove that

$$d = \sqrt{au + bv + cw}.$$

4°. Express the length d of the diagonal $\overrightarrow{OD}$ in terms of the angles α, β, γ and the coordinates u, v, w of the projections of the directed line segment $\overrightarrow{OD}$ on the axes $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$.

5°. Express the volume V of the given parallelepiped in terms of $a, b, c, \alpha, \beta, \gamma$.

6°. Two rays $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ emanate from point O; the first ray forms with the axes $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ the angles $\alpha_1, \beta_1, \gamma_1$; the second ray forms the angles $\alpha_2, \beta_2, \gamma_2$. Express the cosine of the angle θ between the rays $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in terms of $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$.

7°. Find the shortest distance between the straight lines OD and AB if $a, b, c, \alpha, \beta, \gamma$ are given.

8°. Find the distance δ between point D and straight line AB.

Solution. 1°. Consider the vectors $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, $\mathbf{c} = \overrightarrow{OC}$, $\mathbf{d} = \overrightarrow{OD}$. Then

$$\mathbf{d} = \mathbf{a} + \mathbf{b} + \mathbf{c}$$

whence

$$d = \sqrt{\mathbf{d}^2} = \sqrt{(\mathbf{a} + \mathbf{b} + \mathbf{c})^2} = \sqrt{a^2 + b^2 + c^2 + 2bc \cos \alpha + 2ca \cos \beta + 2ab \cos \gamma}.$$

2°.

$$\cos \varphi_1 = \frac{\mathbf{ad}}{|\mathbf{a}|\,|\mathbf{d}|} = \frac{\mathbf{a}(\mathbf{a} + \mathbf{b} + \mathbf{c})}{|\mathbf{a}|\,|\mathbf{d}|} = \frac{a^2 + ab \cos \gamma + ac \cos \beta}{ad} = \frac{a + b \cos \gamma + c \cos \beta}{d}.$$

The formulas for $\cos \varphi_2$ and $\cos \varphi_3$ are derived in similar fashion. Thus

$$\cos \varphi_1 = \frac{a + b \cos \gamma + c \cos \beta}{d},$$

$$\cos \varphi_2 = \frac{a \cos \gamma + b + c \cos \alpha}{d},$$

$$\cos \varphi_3 = \frac{a \cos \beta + b \cos \alpha + c}{d}.$$

3°. From the formulas of item 2° we have

$$u = d \cos \varphi_1 = a + b \cos \gamma + c \cos \beta,$$

$$v = d \cos \varphi_2 = a \cos \gamma + b + c \cos \alpha,$$

$$w = d \cos \varphi_3 = a \cos \beta + b \cos \alpha + c.$$

Multiplying both sides of each of these formulas by a, b, c respectively and adding the resulting formulas termwise, we get

$$au + bv + cw = a(a + b\cos\gamma + c\cos\beta) + b(a\cos\gamma + b + c\cos\alpha) + c(a\cos\beta + b\cos\alpha + c) = d^2.$$

4°. From the formulas of item 2° it follows that

$$a + b\cos\gamma + c\cos\beta = u,$$
$$a\cos\gamma + b + c\cos\alpha = v,$$
$$a\cos\beta + b\cos\alpha + c = w.$$

Solving this system for a, b, c, we obtain

$$a = \frac{1}{\delta}\begin{vmatrix} u & \cos\gamma & \cos\beta \\ v & 1 & \cos\alpha \\ w & \cos\alpha & 1 \end{vmatrix},$$

$$b = \frac{1}{\delta}\begin{vmatrix} 1 & u & \cos\beta \\ \cos\gamma & v & \cos\alpha \\ \cos\beta & w & 1 \end{vmatrix},$$

$$c = \frac{1}{\delta}\begin{vmatrix} 1 & \cos\gamma & u \\ \cos\gamma & 1 & v \\ \cos\beta & \cos\alpha & w \end{vmatrix},$$

where

$$\delta = \begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix} = 1 + 2\cos\alpha\cos\beta\cos\gamma - \cos^2\alpha - \cos^2\beta - \cos^2\gamma,$$

and, consequently, on the basis of the formula

$$d^2 = au + bv + cw$$

we have

$$d^2 = \frac{1}{\delta}\left(u\begin{vmatrix} u & \cos\gamma & \cos\beta \\ v & 1 & \cos\alpha \\ w & \cos\alpha & 1 \end{vmatrix} + v\begin{vmatrix} 1 & u & \cos\beta \\ \cos\gamma & v & \cos\alpha \\ \cos\beta & w & 1 \end{vmatrix} + w\begin{vmatrix} 1 & \cos\gamma & u \\ \cos\gamma & 1 & v \\ \cos\beta & \cos\alpha & w \end{vmatrix}\right)$$

$$= -\frac{1}{\delta}\begin{vmatrix} 1 & \cos\gamma & \cos\beta & u \\ \cos\gamma & 1 & \cos\alpha & v \\ \cos\beta & \cos\alpha & 1 & w \\ u & v & w & 0 \end{vmatrix},$$

so that

$$d = \frac{1}{\sqrt{\delta}}\left(-\begin{vmatrix} 1 & \cos\gamma & \cos\beta & u \\ \cos\gamma & 1 & \cos\alpha & v \\ \cos\beta & \cos\alpha & 1 & w \\ u & v & w & 0 \end{vmatrix}\right)^{1/2},$$

5°. $V^2 = (\mathbf{a}, \mathbf{b}, \mathbf{c})^2 = \begin{vmatrix} \mathbf{a}^2 & \mathbf{ab} & \mathbf{ac} \\ \mathbf{ba} & \mathbf{b}^2 & \mathbf{bc} \\ \mathbf{ca} & \mathbf{cb} & \mathbf{c}^2 \end{vmatrix}$

$$= \begin{vmatrix} a^2 & ab\cos\gamma & ac\cos\beta \\ ab\cos\gamma & b^2 & bc\cos\alpha \\ ac\cos\beta & bc\cos\alpha & c^2 \end{vmatrix} = a^2b^2c^2 \begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix},$$

and so

$$V = abc\sqrt{\delta} = abc\sqrt{1 + 2\cos\alpha\cos\beta\cos\gamma - \cos^2\alpha - \cos^2\beta - \cos^2\gamma}.$$

6°. Let us consider the vectors $\mathbf{p} = \overrightarrow{OP}$, $\mathbf{q} = \overrightarrow{OQ}$, assuming that $|\mathbf{p}| = |\mathbf{q}| = 1$.

Expand the vector $\mathbf{p}$ in terms of the basis $\mathbf{a}, \mathbf{b}, \mathbf{c}$:

$$\mathbf{p} = p_1\mathbf{a} + p_2\mathbf{b} + p_3\mathbf{c}.$$

Taking the scalar product of both sides of this equation into $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in succession, we obtain

$$\cos\alpha_1 = p_1 + p_2\cos\gamma - p_3\cos\beta,$$

$$\cos\beta_1 = p_1\cos\gamma + p_2 + p_3\cos\alpha,$$

$$\cos\gamma_1 = p_1\cos\beta + p_2\cos\alpha + p_3,$$

whence

$$p_1 = \frac{1}{\delta}\begin{vmatrix} \cos\alpha_1 & \cos\gamma & \cos\beta \\ \cos\beta_1 & 1 & \cos\alpha \\ \cos\gamma_1 & \cos\alpha & 1 \end{vmatrix},$$

$$p_2 = \frac{1}{\delta}\begin{vmatrix} 1 & \cos\alpha_1 & \cos\beta \\ \cos\gamma & \cos\beta_1 & \cos\alpha \\ \cos\beta & \cos\gamma_1 & 1 \end{vmatrix},$$

$$p_3 = \frac{1}{\delta}\begin{vmatrix} 1 & \cos\gamma & \cos\alpha_1 \\ \cos\gamma & 1 & \cos\beta_1 \\ \cos\beta & \cos\alpha & \cos\gamma_1 \end{vmatrix}.$$

Furthermore, let $\mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*$ be the reciprocal basis of $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Expand the vector $\mathbf{q}$ in terms of the basis $\mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*$:

$$\mathbf{q} = q_1\mathbf{a}^* + q_2\mathbf{b}^* + q_3\mathbf{c}^*.$$

Taking the scalar product of both sides of this relation by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in succession, we obtain

$$q_1 = \cos\alpha_2, \quad q_2 = \cos\beta_2, \quad q_3 = \cos\gamma_2.$$

We get

$$\mathbf{q} = \cos\alpha_2\mathbf{a}^* + \cos\beta_2\mathbf{b}^* + \cos\gamma_2\mathbf{c}^*.$$

We now find

$$\cos(\mathbf{p}, \mathbf{q}) = \cos\theta = \mathbf{pq} = (p_1\mathbf{a} + p_2\mathbf{b} + p_3\mathbf{c})(q_1\mathbf{a}^* + q_2\mathbf{b}^* + q_3\mathbf{c}^*)$$
$$= p_1q_1 + p_2q_2 + p_3q_3$$
$$= \frac{1}{\delta}\left(\cos\alpha_2 \begin{vmatrix} \cos\alpha_1 & \cos\gamma & \cos\beta \\ \cos\beta_1 & 1 & \cos\alpha \\ \cos\gamma_1 & \cos\alpha & 1 \end{vmatrix} + \cos\beta_2 \begin{vmatrix} 1 & \cos\alpha_1 & \cos\beta \\ \cos\gamma & \cos\beta_1 & \cos\alpha \\ \cos\beta & \cos\gamma_1 & 1 \end{vmatrix}\right.$$
$$\left. + \cos\gamma_2 \begin{vmatrix} 1 & \cos\gamma & \cos\alpha_1 \\ \cos\gamma & 1 & \cos\beta_1 \\ \cos\beta & \cos\alpha & \cos\gamma_1 \end{vmatrix}\right) = -\frac{1}{\delta}\begin{vmatrix} 1 & \cos\gamma & \cos\beta & \cos\alpha_1 \\ \cos\gamma & 1 & \cos\alpha & \cos\beta_1 \\ \cos\beta & \cos\alpha & 1 & \cos\gamma_1 \\ \cos\alpha & \cos\beta_2 & \cos\gamma_2 & 0 \end{vmatrix}.$$

The formula for $\cos\theta$ can be written more compactly:

$$\begin{vmatrix} 1 & \cos\gamma & \cos\beta & \cos\alpha_1 \\ \cos\gamma & 1 & \cos\alpha & \cos\beta_1 \\ \cos\beta & \cos\alpha & 1 & \cos\gamma_1 \\ \cos\alpha_2 & \cos\beta_2 & \cos\gamma_2 & \cos\theta \end{vmatrix} = 0.$$

7°. The shortest distance d between two noncollinear straight lines in space is equal to the length of the projection of any line segment M_1M_2, whose ends lie on the given lines, onto the common perpendicular to the given straight lines.

Thus, in item 7°

$$M_1 = O, \quad M_2 = B, \quad \overrightarrow{M_1M_2} = \overrightarrow{OB} = \mathbf{b}.$$

Now, the direction of the common perpendicular to the straight lines OD and AB is given by the vector product of the vector $\mathbf{a} + \mathbf{b} + \mathbf{c} = \overrightarrow{OD}$ into the vector $\mathbf{b} - \mathbf{a} = \overrightarrow{AB}$:

$$[\mathbf{a} + \mathbf{b} + \mathbf{c}, \mathbf{b} - \mathbf{a}] = 2[\mathbf{a}, \mathbf{b}] + [\mathbf{c}, \mathbf{b}] + [\mathbf{a}, \mathbf{c}].$$

Thus,

$$d = \frac{|\mathbf{b}(2[\mathbf{a}, \mathbf{b}] + [\mathbf{c}, \mathbf{b}] + [\mathbf{a}, \mathbf{c}])|}{|2[\mathbf{a}, \mathbf{b}] + [\mathbf{c}, \mathbf{b}] + [\mathbf{a}, \mathbf{c}]|} = \frac{|\mathbf{a}, \mathbf{b}, \mathbf{c}|}{\sqrt{T}}$$

where (see item 5°)

$$|\mathbf{a}, \mathbf{b}, \mathbf{c}| = abc\sqrt{\delta},$$

$$\begin{aligned}T = 4[\mathbf{a}, \mathbf{b}]^2 + [\mathbf{c}, \mathbf{b}]^2 + [\mathbf{a}, \mathbf{c}]^2 + 4([\mathbf{a}, \mathbf{b}]\cdot[\mathbf{c}, \mathbf{b}]) + 2([\mathbf{c}, \mathbf{b}]\cdot[\mathbf{a}, \mathbf{c}])\\ + 4([\mathbf{a}, \mathbf{b}]\cdot[\mathbf{a}, \mathbf{c}]) = 4a^2b^2\sin^2\gamma + c^2b^2\sin^2\alpha + a^2c^2\sin^2\beta\\ + 4b^2ac(\cos\beta - \cos\alpha\cos\gamma) + 4a^2bc(\cos\alpha - \cos\beta\cos\gamma)\\ -2c^2ab(\cos\gamma - \cos\alpha\cos\beta).\end{aligned}$$

And so $d = abc\dfrac{\sqrt{\delta}}{\sqrt{T}}$.

8°. Suppose $\mathbf{e}$ is the direction vector of the straight line l and B is an arbitrary point in space. Take some point A on l and let $\overrightarrow{AB} = \mathbf{a}$. Then the distance d from point B to l is found from the formula $d = |[\mathbf{a}, \mathbf{e}]|/|\mathbf{e}|$. Indeed, $|[\mathbf{a}, \mathbf{e}]| = |\mathbf{a}|\,|\mathbf{e}|\sin\varphi = |\mathbf{e}|\,d$.

In particular, if $\mathbf{e}$ is the unit vector, then

$$d = |[\mathbf{a}, \mathbf{e}]|.$$

In determining the distance from point D to the straight line AB, note that the direction vector of that line is equal to $\mathbf{a} - \mathbf{b}$, and since $\overrightarrow{AD} = \mathbf{b} + \mathbf{c}$, it follows, using the formula

$$d = \frac{|[\mathbf{a}, \mathbf{e}]|}{|\mathbf{e}|},$$

that

$$d = \frac{|[(\mathbf{a} - \mathbf{b}), (\mathbf{b} + \mathbf{c})]|}{|\mathbf{a} - \mathbf{b}|} = \frac{|[\mathbf{a}, \mathbf{b}] + [\mathbf{a}, \mathbf{c}] + [\mathbf{c}, \mathbf{b}]|}{|\mathbf{a} - \mathbf{b}|} = \frac{\sqrt{T_1}}{\sqrt{T_2}}$$

where

$$\begin{aligned}T_1 = a^2b^2\sin^2\gamma + b^2c^2\sin^2\alpha + c^2a^2\sin^2\beta\\ + 2a^2bc(\cos\alpha - \cos\beta\cos\gamma) + 2b^2ca(\cos\beta - \cos\gamma\cos\alpha)\\ - 2c^2ab(\cos\gamma - \cos\alpha\cos\beta),\end{aligned}$$

$$T_2 = a^2 + b^2 - 2ab\cos\gamma.$$

Problem 3. Given the plane angles $\angle BOC = a$, $\angle COA = b$, $\angle AOB = c$ of the trihedral angle $OABC$. The ray l emanates from point O and forms with the edges $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ of the given trihedral angle the equal angles φ. Find $\tan\varphi$.

Solution. On the edges of the given trihedral angle, choose points A, B, C so that $OA = OB = OC = 1$ and then consider the unit vectors $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, $\mathbf{c} = \overrightarrow{OC}$.

Let $\overrightarrow{OP}$ be a directed line segment of length 1 such that $\angle POA = \angle POB = \angle POC = \varphi$. We now consider the unit vector $\mathbf{x} = \overrightarrow{OP}$. Then

$$|\mathbf{xa} = \mathbf{xb} = \mathbf{xc}| = \cos\varphi$$

and, by the Gibbs formula,

$$\mathbf{x} = (\mathbf{xa})\cdot\mathbf{a}^* + (\mathbf{xb})\cdot\mathbf{b}^* + (\mathbf{xc})\cdot\mathbf{c}^* = (\mathbf{a}^* + \mathbf{b}^* + \mathbf{c}^*)\cos\varphi \qquad (*)$$

where $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ is the reciprocal (or dual) basis of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$; that is,

$$\mathbf{a}^* = \frac{[\mathbf{b}, \mathbf{c}]}{(\mathbf{a}, \mathbf{b}, \mathbf{c})}, \quad \mathbf{b}^* = \frac{[\mathbf{c}, \mathbf{a}]}{(\mathbf{a}, \mathbf{b}, \mathbf{c})}, \quad \mathbf{c}^* = \frac{[\mathbf{a}, \mathbf{b}]}{(\mathbf{a}, \mathbf{b}, \mathbf{c})}.$$

Raising both sides of (*) to a scalar square, we obtain

$$1 = \cos^2\varphi(\mathbf{a}^* + \mathbf{b}^* + \mathbf{c}^*)^2$$

whence

$$1 + \tan^2\varphi = \mathbf{a}^{*2} + \mathbf{b}^{*2} + \mathbf{c}^{*2} + 2\mathbf{b}^*\mathbf{c}^* + 2\mathbf{c}^*\mathbf{a}^* + 2\mathbf{a}^*\mathbf{b}^* = \frac{T_1}{T_2}$$

where

$$T_1 = \sin^2 a + \sin^2 b + \sin^2 c + 2(\cos b\cos c - \cos a) + 2(\cos c\cos a - \cos b) + 2(\cos a\cos b - \cos c),$$

$$T_2 = (\mathbf{a}, \mathbf{b}, \mathbf{c})^2 = 1 + 2\cos a\cos b\cos c - \cos^2 a - \cos^2 b - \cos^2 c.$$

Thus

$$\tan^2\varphi = \frac{T_1}{T_2} - 1 = \frac{T_1 - T_2}{T_2},$$

$$\begin{aligned} T_1 - T_2 &= \sin^2 a + \sin^2 b + \sin^2 c + 2(\cos b\cos c - \cos a) \\ &\quad + 2(\cos c\cos a - \cos b) + 2(\cos a\cos b - \cos c) \\ &\quad - 1 - 2\cos a\cos b\cos c + \cos^2 a + \cos^2 b + \cos^2 c \\ &= 2[1 - (\cos a + \cos b + \cos c) + (\cos b\cos c + \cos c\cos a + \cos a\cos b) \\ &\quad - \cos a\cos b\cos c] \\ &= 2(1 - \cos a)(1 - \cos b)(1 - \cos c) = 16\sin^2\frac{a}{2}\sin^2\frac{b}{2}\sin^2\frac{c}{2}, \end{aligned}$$

$$\tan^2\varphi = \frac{T_1 - T_2}{T_2} = \frac{16\sin^2\dfrac{a}{2}\sin^2\dfrac{b}{2}\sin^2\dfrac{c}{2}}{\begin{vmatrix} 1 & \cos b & \cos c \\ \cos b & 1 & \cos a \\ \cos c & \cos a & 1 \end{vmatrix}}.$$

The angle φ may be regarded as an acute angle (because if φ is obtuse the direction of the line segment $\overrightarrow{OP}$ may be reversed).

Thus

$$\tan\varphi = \frac{4\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}}{\Delta}$$

where

$$\Delta = \begin{vmatrix} 1 & \cos b & \cos c \\ \cos b & 1 & \cos a \\ \cos c & \cos a & 1 \end{vmatrix}$$

$$= 1 + 2\cos a\cos b\cos c - \cos^2 a - \cos^2 b - \cos^2 c.$$

Problem 4. Given in a tetrahedron $OABC$ the lengths of the edges $OA = a$, $OB = b$, $OC = c$ and the plane angles $\angle BOC = \alpha$, $\angle COA = \beta$, $\angle AOB = \gamma$. Let PQ be the common perpendicular to the straight lines OA and BC (point P lies on line OA, point Q on line BC). Find the ratios

$$\frac{\overrightarrow{OP}}{\overrightarrow{OA}} = \lambda, \quad \frac{\overrightarrow{BQ}}{\overrightarrow{BC}} = \mu.$$

Solution. Consider the vectors $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, $\mathbf{c} = \overrightarrow{OC}$, $\mathbf{p} = \overrightarrow{OP}$, $\mathbf{q} = \overrightarrow{OQ}$, $\mathbf{t} = \overrightarrow{BQ}$, $\mathbf{s} = \overrightarrow{PQ}$. Then

$$\mathbf{p} = \lambda\mathbf{a}, \quad \mathbf{t} = \mu(\mathbf{c} - \mathbf{b}).$$

Furthermore,

$$\mathbf{p} + \mathbf{s} - \mathbf{t} - \mathbf{b} = \mathbf{0}$$

whence

$$\mathbf{s} = -\lambda\mathbf{a} + \mu(\mathbf{c} - \mathbf{b}) + \mathbf{b}.$$

Since the vector $\mathbf{s}$ is perpendicular to the vectors $\mathbf{a}$ and $\mathbf{b} - \mathbf{c}$, we have

$$\mathbf{as} = 0, \quad (\mathbf{b} - \mathbf{c})\,\mathbf{s} = 0$$

or

$$\mathbf{a}(-\lambda\mathbf{a} + \mu(\mathbf{c} - \mathbf{b}) + \mathbf{b}) = 0,$$
$$(\mathbf{b} - \mathbf{c})\,(-\lambda\mathbf{a} + \mu(\mathbf{c} - \mathbf{b}) + \mathbf{b}) = 0$$

or

$$\lambda a^2 - \mu\mathbf{a}(\mathbf{c} - \mathbf{b}) = \mathbf{ab},$$
$$-\lambda\mathbf{a}(\mathbf{c} - \mathbf{b}) + \mu(\mathbf{c} - \mathbf{b})^2 = -\mathbf{b}(\mathbf{c} - \mathbf{b}).$$

From the resulting system of equations that are linear in λ and μ, we obtain these unknowns:

$$\lambda = \frac{\overrightarrow{OP}}{\overrightarrow{OA}} = \frac{b}{a}\,\frac{\cos\gamma(c^2+b^2-2bc\cos\alpha)-(c\cos\beta-b\cos\gamma)(c\cos\alpha-b)}{c^2\sin^2\beta+b^2\sin^2\gamma-2bc(\cos\alpha-\cos\beta\cos\gamma)},$$

$$\mu = \frac{\overrightarrow{BQ}}{\overrightarrow{BC}} = b\,\frac{-c\cos\alpha+b+\cos\gamma(c\cos\beta-b\cos\gamma)}{c^2\sin^2\beta+b^2\sin^2\gamma-2bc(\cos\alpha-\cos\beta\cos\gamma)}.$$

Problem 5. We give here a vectorial derivation of the basic formulas used in the theory of *axonometry* (axonometric projections).

Suppose Ox, Oy, Oz are three pairwise perpendicular axes. Let us consider the plane π that intersects these axes in the points A, B, C respectively. We assume the space to be oriented by the ordered triple of axes Ox, Oy, Oz. Denote by O' the orthogonal projection of point O on the plane $\pi = ABC$, by $\alpha_1, \alpha_2, \alpha_3$ the angles of the axes Ox, Oy, Oz with the plane π, and by $\beta_1, \beta_2, \beta_3$ the respective angles $BO'C$, $CO'A$, $AO'B$.

Prove that

1°.
$$\cos\beta_1 = -\tan\alpha_2\tan\alpha_3,$$
$$\cos\beta_2 = -\tan\alpha_3\tan\alpha_1,$$
$$\cos\beta_3 = -\tan\alpha_1\tan\alpha_2.$$

2°. $$\frac{\sin\beta_1}{\sin\alpha_1\cos\alpha_1} = \frac{\sin\beta_2}{\sin\alpha_2\cos\alpha_2} = \frac{\sin\beta_3}{\sin\alpha_3\cos\alpha_3} = \frac{1}{\cos\alpha_1\cos\alpha_2\cos\alpha_3}.$$

3°. $$\frac{\sin 2\beta_1}{\cos^2\alpha_1} = \frac{\sin 2\beta_2}{\cos^2\alpha_2} = \frac{\sin 2\beta_3}{\cos^2\alpha_3} = -2\,\frac{\sin\alpha_1\sin\alpha_2\sin\alpha_3}{\cos^2\alpha_1\cos^2\alpha_2\cos^2\alpha_3}.$$

4°. There exists a triangle, the lengths of whose sides are proportional to $\cos^2\alpha_1$, $\cos^2\alpha_2$, $\cos^2\alpha_3$; one of these triangles is a triangle formed by the feet of the altitudes of the triangle ABC (*Schlömilch's theorem*).

Solution. 1°.

$$\overrightarrow{OB}\cdot\overrightarrow{OC} = (\overrightarrow{OO'}+\overrightarrow{O'B})(\overrightarrow{OO'}+\overrightarrow{O'C}) = \overrightarrow{OO'}^2 + \overrightarrow{O'B}\cdot\overrightarrow{O'C} = 0 \quad (*)$$

since $\overrightarrow{OO'}\perp\overrightarrow{O'C}$, $\overrightarrow{OO'}\perp\overrightarrow{O'B}$, $\overrightarrow{OB}\perp\overrightarrow{OC}$; consequently, from the relation (*) we obtain

$$O'B\tan\alpha_2\cdot O'C\tan\alpha_3 + O'B\cdot O'C\cos\beta_1 = 0$$

whence

$$\cos\beta_1 = -\tan\alpha_2\tan\alpha_3.$$

The formulas

$$\cos\beta_2 = -\tan\alpha_3\tan\alpha_1,$$
$$\cos\beta_3 = -\tan\alpha_1\tan\alpha_2$$

are derived in similar fashion.

2°. $[\overrightarrow{O'B}, \overrightarrow{O'C}] = [\overrightarrow{O'O} + \overrightarrow{OB}, \overrightarrow{O'O} + \overrightarrow{OC}]$

$$= [\overrightarrow{O'O}, \overrightarrow{OC}] + [\overrightarrow{OB}, \overrightarrow{O'O}] + [\overrightarrow{OB}, \overrightarrow{OC}]$$

whence

$$[\overrightarrow{O'B}, \overrightarrow{O'C}]\cdot\overrightarrow{OO'} = [\overrightarrow{OB}, \overrightarrow{OC}]\cdot\overrightarrow{OO'}. \qquad (**)$$

Furthermore,

$$[\overrightarrow{O'B}, \overrightarrow{O'C}]\cdot\overrightarrow{OO'} = |[\overrightarrow{O'B}, \overrightarrow{O'C}]|\, OO'$$

$$= O'B\cdot O'C \sin\beta_1\cdot OO' = OB\cos\alpha_2\cdot OC\cos\alpha_3\cdot OO'\sin\beta_1,$$

$$[\overrightarrow{OB}, \overrightarrow{OC}]\cdot\overrightarrow{OO'} = OB\cdot OC\cdot OO'\cos\left(\frac{\pi}{2} - \alpha_1\right) = OB\cdot OC\cdot OO'\sin\alpha_1$$

and, hence, by virtue of (**) we have

$$\sin\beta_1\cos\alpha_2\cos\alpha_3 = \sin\alpha_1.$$

Consequently,

$$\frac{\sin\beta_1}{\sin\alpha_1\cos\alpha_1} = \frac{1}{\cos\alpha_1\cos\alpha_2\cos\alpha_3}.$$

In similar fashion, proof is given that

$$\frac{\sin\beta_2}{\sin\alpha_2\cos\alpha_2} = \frac{\sin\beta_3}{\sin\alpha_3\cos\alpha_3} = \frac{1}{\cos\alpha_1\cos\alpha_2\cos\alpha_3}.$$

3°. The solution follows from items 1° and 2°.

4°. Rewriting the relations of item 3° in the form

$$\frac{\cos^2\alpha_1}{\sin 2\left(\beta_1 - \frac{\pi}{2}\right)} = \frac{\cos^2\alpha_2}{\sin 2\left(\beta - \frac{\pi}{2}\right)} = \frac{\cos^2\alpha_3}{\sin 2\left(\beta_3 - \frac{\pi}{2}\right)}$$

and noting that

$$2\left(\beta_1 - \frac{\pi}{2}\right) + 2\left(\beta_2 - \frac{\pi}{2}\right) + 2\left(\beta_3 - \frac{\pi}{2}\right) = \pi,$$

we conclude that there exists a triangle Δ with angles $2\left(\beta_1 - \frac{\pi}{2}\right)$, $2\left(\beta_2 - \frac{\pi}{2}\right)$, $2\left(\beta_3 - \frac{\pi}{2}\right)$. Indeed,

$$\beta_1 > \pi/2, \quad \beta_2 > \pi/2, \quad \beta_3 > \pi/2$$

since all plane angles of the trihedral angle $OABC$ are equal to $\pi/2$ and therefore $\beta_1, \beta_2, \beta_3$ are obtuse angles: a sphere with diameter AB passes through the point O ($\angle AOB = \pi/2$), the plane ABC intersects the sphere along a great circle since AB is the diameter of the sphere, the projection O' of point O on the plane ABC will be inside that circle and, hence, the angle $AO'B = \beta_1$ is obtuse (similarly $\pi/2 < \beta_2 < \pi$, $\pi/2 < \beta_3 < \pi$).

From these relations it follows that $\cos^2\alpha_1, \cos^2\alpha_2, \cos^2\alpha_3$ are proportional to the sines of the angles of the triangle Δ, and so also to the lengths of its sides.

Finally, we will now prove that one of these triangles is formed by the feet of the altitudes of $\triangle ABC$. Indeed, ABC is an acute-angled triangle because the lengths of its sides

$$AB = \sqrt{OA^2 + OB^2}, \quad BC = \sqrt{OB^2 + OC^2}, \quad CA = \sqrt{OC^2 + OA^2}$$

and, hence, for example, $AB^2 + BC^2 > AC^2$ (angle B is acute) and so forth. Furthermore, since $OA \perp OBC$, it follows that $OA \perp BC$ and, hence, on the basis of the theorem of three perpendiculars $AO' \perp BC$ and, similarly, $BO' \perp CA$, $CO' \perp AB$, that is O' is the point of intersection of the altitudes of $\triangle ABC$. Let A_1, B_1, C_1 be the feet of the altitudes of $\triangle ABC$. What we then have is that since $\angle AC_1O' = \angle AB_1O' = \pi/2$, it follows that points A, C_1, O', B_1 lie on one circle (with diameter AO'). From this it follows that $\angle C_1B_1O' = \angle C_1AO'$ (both angles are intercepted by the arc C_1O'). But $\angle C_1AO' = \frac{\pi}{2} - B$ and so $\angle C_1B_1O' = \frac{\pi}{2} - B$. Similarly, the points B_1, O', A_1, C lie on one circle, $\angle A_1B_1O' = \angle A_1CO' = \frac{\pi}{2} - B$. Thus, B_1O' is the bisector of the interior angle B_1 of $\triangle A_1B_1C_1$. Now, from $\triangle O'B_1C_1$ we have

$$\frac{B_1}{2} + \frac{C_1}{2} + \beta_1 = \pi$$

and since

$$A_1 + B_1 + C_1 = \pi$$

it follows that

$$A_1 = 2\beta_1 - \pi = 2\left(\beta_1 - \frac{\pi}{2}\right).$$

Similarly,

$$B_1 = 2\left(\beta_2 - \frac{\pi}{2}\right), \quad C_1 = 2\left(\beta_3 - \frac{\pi}{2}\right).$$

Problem 6. On the base of the polyhedron $OACBPQ$ lies a rectangle $OACB$. The edge PQ is parallel to the plane of the rectangle, and the

orthogonal projections P_1 and Q_1 of points P and Q on the plane $OABC$ are such that

$$P_1O = P_1A = Q_1B = Q_1C$$

(this kind of polyhedron is called a *wedge*).

Given the lengths of the sides of the rectangle $OACB$:

$$OA = a, \qquad OB = b,$$

the length c of segment PQ and the altitude $h = PP_1 = QQ_1$. Compute the cosine of the dihedral angle $A(CQ)\,B = \varphi$.

Solution. Introduce a rectangular Cartesian system of coordinates $Oxyz$, choosing the directions of the x- and y-axes along $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. The coordinates of the vertices of the wedge in the chosen system are:

$$O = (0, 0, 0), \quad A = (a, 0, 0), \quad C = (a, b, 0), \quad B = (0, b, 0),$$

$$P = \left(\frac{a}{2}, \frac{b-c}{2}, h\right), \quad Q = \left(\frac{a}{2}, \frac{b+c}{2}, h\right).$$

From this we find the vectors

$$\overrightarrow{CA} = \{0, -b, 0\}\uparrow\uparrow\{0, -1, 0\} = \mathbf{x},$$

$$\overrightarrow{CQ} = \left\{-\frac{a}{2}\cdot \frac{c-b}{2}, h\right\}\uparrow\uparrow\{-a, c-b, 2h\} = \mathbf{y},$$

$$\overrightarrow{CB} = \{-a, 0, 0\}\uparrow\uparrow\{-1, 0, 0\} = \mathbf{z}.$$

We also find

$$[\mathbf{y}, \mathbf{x}] = \left\{\begin{vmatrix} c-b & 2h \\ -1 & 0 \end{vmatrix}, \begin{vmatrix} 2h & -a \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} -a & c-b \\ 0 & -1 \end{vmatrix}\right\} = \{2h,\ 0,\ a\},$$

$$[\mathbf{y}, \mathbf{z}] = \left\{\begin{vmatrix} c-b & 2h \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 2h & -a \\ 0 & -1 \end{vmatrix}, \begin{vmatrix} -a & c-b \\ -1 & 0 \end{vmatrix}\right\} = \{0,\ -2h,\ c-b\}$$

and, consequently, we have

$$\cos\varphi = \frac{[\mathbf{y}, \mathbf{x}]\cdot[\mathbf{y}, \mathbf{z}]}{|[\mathbf{y}, \mathbf{x}]|\,|[\mathbf{y}, \mathbf{z}]|} = \frac{a(c-b)}{\sqrt{4h^2 + a^2}\sqrt{4h^2 + (c-b)^2}}.$$

Sec. 3. Vectors in the plane and in space (problems with hints and answers)

1. Given two vectors $\overrightarrow{CA} = \mathbf{b}$, $\overrightarrow{CB} = \mathbf{a} \neq 0$. Let P be the orthogonal projection of point A on the straight line BC. Find the vector $\overrightarrow{CP}$.

Answer. $\dfrac{(\mathbf{ab})\,\mathbf{a}}{a^2}$.

2. Four points A, B, C, D are located (in the plane or in space) so that $AD \perp BC$, $BD \perp CA$. Prove that $CD \perp AB$.

Hint. Assuming $\overrightarrow{DA} = \mathbf{r}_1$, $\overrightarrow{DB} = \mathbf{r}_2$, $\overrightarrow{DC} = \mathbf{r}_3$, we find $\mathbf{r}_1(\mathbf{r}_2 - \mathbf{r}_3) = 0$, $\mathbf{r}_2(\mathbf{r}_3 - \mathbf{r}_1) = 0$, whence $\mathbf{r}_3(\mathbf{r}_1 - \mathbf{r}_2) = 0$.

3. An arbitrary point O is joined to the centroid G of $\triangle ABC$ and a point P is constructed such that $\overrightarrow{OP} = 3\overrightarrow{OG}$. Let A', B', C' be points symmetric to the point P about the points A, B, C. Prove that O is the centre of gravity of the four points A', B', C', P.

4. Let ABC be an arbitrary triangle. We consider 12 vectors whose initial points are the centre of an inscribed circle and the centres of escribed circles, and the end points are the points of contact of circles and the sides of the triangle. Prove that the sum of these vectors is equal to the vector $\overrightarrow{OH}$, where O is the centre of the circle (ABC), and H is the orthocentre of the triangle ABC.

5. ABC is an arbitrary nondegenerate triangle lying on an oriented plane; P is an arbitrary point lying in the plane of the triangle. The vectors $\overrightarrow{PA}, \overrightarrow{PB}, \overrightarrow{PC}$ are coplanar and, hence, are linearly dependent:

$$\alpha\overrightarrow{PA} + \beta\overrightarrow{PB} + \gamma\overrightarrow{PC} = \mathbf{0}$$

(at least one of the numbers α, β, γ is different from zero). Prove that

$$\alpha : \beta : \gamma = (PBC) : (PCA) : (PAB).$$

Hint. Project the zero vector $\alpha\overrightarrow{PA} + \beta\overrightarrow{PB} + \gamma\overrightarrow{PC}$ onto the straight line BC in the direction of the straight line PA. Let P_1 be the point into which are projected the points P and A. Then

$$\beta\overrightarrow{P_1B} + \gamma\overrightarrow{P_1C} = \mathbf{0}.$$

6. Let I be the centre of a circle (I) inscribed in a triangle ABC; let D, E, F be the points of contact of the circle (I) with the sides BC, CA, AB; let P be an arbitrary point lying in the plane of $\triangle ABC$; let P_a, P_b, P_c be orthogonal projections of point P on the sides BC, CA, AB. Prove that the circle passing through the centroids of the triangles EP_aF, FP_bD, DP_cE has a diameter equal to $\frac{1}{3}IP$.

Hint. Let G be the centroid of the triangle DEF, and let G_a, G_b, G_c be the centroids of the triangles EP_aF, FP_bD, DP_cE; then

$$\overrightarrow{IG_a} = \frac{1}{3}(\overrightarrow{IE} + \overrightarrow{IF} + \overrightarrow{IP_a})$$

$$= \frac{1}{3}(\overrightarrow{IE} + \overrightarrow{IF} + \overrightarrow{ID} + \overrightarrow{DP_a}) = \overrightarrow{IG} + \frac{1}{3}\overrightarrow{DP_a}.$$

Thus, the point G_a is constructed in the following manner: from point G lay off a directed line segment equal to one third of the projection $\overrightarrow{IP}$ on the side BC. From this it immediately follows that the points G_a, G_b, G_c lie on the circle whose diameter is obtained by laying off from point G a vector determined by the directed segment $\frac{1}{3}\overrightarrow{IP}$.

7. ABC is an arbitrary triangle; A_1, B_1, C_1 are the midpoints of its sides BC, CA, AB; A_2, B_2, C_2 are the feet of its altitudes; O is the centre of the circle (ABC); G is the centroid of $\triangle ABC$ (the point of intersection of its medians); H is the orthocentre (the point of intersection of the altitudes of $\triangle ABC$); A_3, B_3, C_3 are the midpoints of the segments AH, BH, CH; A_4, B_4, C_4 are the second points of intersection of the circle (ABC) with the altitudes AH, BH, CH (Fig. 1). Prove that:

1°. The points O, G, H are collinear and $\overrightarrow{OG} = \frac{1}{2}\overrightarrow{GH}$.

2°. The points $A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3$ lie on one circle called the *Euler circle* of $\triangle ABC$ or the nine-point circle (O_9) of $\triangle ABC$.

3°. The points A_4, B_4, C_4 are symmetric to the orthocentre H with respect to the straight lines BC, CA, AB.

Hint. This problem may be solved vectorially. First prove that if O is the centre of the circle (O) circumscribed about $\triangle ABC$, then $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OH}$. (The vectorial solution is left to the reader to carry out; below we give another proof of all these propositions by resorting to complex numbers.)

For the present, note the following simple synthetic solution.

1°. In the case of a homothetic transformation $\left(G, -\frac{1}{2}\right)$, the points A, B, C go into the points A_1, B_1, C_1 and, hence, the altitudes of the triangle ABC (which are now regarded as straight lines) go into the midperpendiculars of the sides BC, CA, AB (the *midperpendicular* of a line segment is a straight line perpendicular to the line segment at its midpoint). From this it follows that the point H of intersection of the altitudes of $\triangle ABC$ goes into the point O of intersection of the midperpendiculars of its sides. Thus, in the case of a homothetic transformation $\left(G, -\frac{1}{2}\right)$, point H goes into point O, which means the points G, H, and O are collinear and $\overrightarrow{OG} = \overrightarrow{GH}/2$.

2°. In the case of a homothetic transformation $\left(G, -\frac{1}{2}\right)$, the circle (ABC) goes into the circle $(A_1B_1C_1)$ and, hence, the centre of the circle $(A_1B_1C_1)$ is the image of the centre O of the circle (ABC) that is, the midpoint O_9

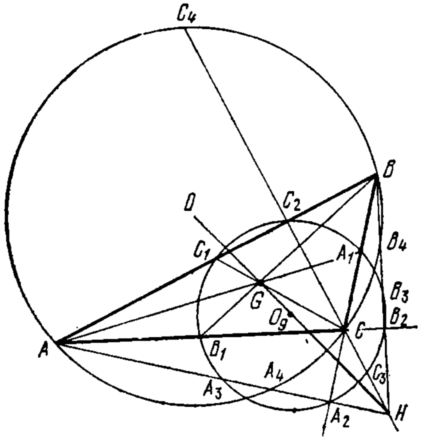

Fig. 1

of segment OH. Since O_9 is the midpoint of segment OH, and the points O and H are projected respectively onto the side BC into the points A_1 and A_2, it follows that the point O_9 is projected into the midpoint of the line segment A_1A_2 and, hence, $O_9'A_1 = O_9'A_2$, where O_9' is the projection of O_9 on the straight line BC. But if the projections of inclined lines are equal so also are equal the inclined lines themselves, hence, $O_9A_1 = O_9A_2$. Similarly, proof is given that $O_9B_1 = O_9B_2$, $O_9C_1 = O_9C_2$ and since $O_9A_1 = O_9B_1 = O_9C_1$ [O_9 is the centre of the circle $(A_1B_1C_1)$], it follows that the circle $(A_1B_1C_1)$ also passes through the points A_2, B_2, C_2. Note that the radius of the circle $(A_1B_1C_1)$ is half the radius R of the circle (ABC).

Now let us consider the homothetic transformation $\left(H, \frac{1}{2}\right)$. Under this transformation, the point O goes into the point O_9, and the circle (ABC) goes into a circle with centre O_9 and radius $R/2$, that is, into the circle $(A_1B_1C_1)$. But under the homothetic transformation $\left(H, \frac{1}{2}\right)$, the points A, B, C go into the points A_3, B_3, C_3 and since the points A, B, C lie on the circle (ABC), it follows that the points A_3, B_3, C_3 lie on the Euler circle $(A_1B_1C_1) = (O_9)$.

3°. Finally, let us consider the homothetic transformation $(H, 2)$. Under this transformation, the circle $(A_1B_1C_1)$ goes into a circle with centre O and radius R, that is, into the circle (ABC). On the other hand, under the transformation $(H, 2)$, the points A_2, B_2, C_2 go into the points A_4, B_4, C_4, which are symmetric to the point H with respect to the straight lines BC, CA, AB. But the points A_2, B_2, C_2 lie on the Euler circle and, hence, the points A_4, B_4, C_4 lie on the circle (ABC).

8. Suppose H, G, O, and O_9 are, respectively, the orthocentre of $\triangle ABC$, its centroid the centre of the circumscribed circle (ABC), and the centre of the Euler circle; let A_1, B_1, C_1 be the midpoints of the sides BC, CA, AB; let A_H, B_H, C_H be the feet of the altitudes; let A', B', C' be points symmetric to the vertices A, B, C of $\triangle ABC$ with respect to its sides BC, CA, AB; let A'', B'', C'' be points in which the altitudes AH, BH, CH intersect the circumscribed circle (ABC); let O' be the centre of the circle $(A'B'C') = (O')$; let α, β, γ be the projections of the point O_9 on the sides BC, CA, AB of $\triangle ABC$. Then:

1°. The triangle $A'B'C'$ is an image of the triangle $\alpha\beta\gamma$ under the homothetic transformation $(G, 4)$.

2°. $4\overrightarrow{O_9\alpha} = \overrightarrow{AA''}$, $4\overrightarrow{O_9\beta} = \overrightarrow{BB''}$, $4\overrightarrow{O_9\gamma} = \overrightarrow{CC''}$.

3°. For the circumscribed circle (ABC) to be tangent with the altitude dropped from A onto BC, it is necessary and sufficient that the centre O_9 of the Euler circle lie on the side BC.

4°. Let O_9' be a point symmetric to the point O_9 with respect to the centre ω of the circle $(\alpha\beta\gamma)$; then the centres O and O' of (ABC) and $(A'B'C')$ are symmetric with respect to the point O_9'.

5°. If the centre O' of the circle $(A'B'C')$ lies inside $\triangle ABC$, then it is a point that has the property that the shortest distances from O' to each vertex A, B, C with a preliminary shift along the straight line to the opposite side are equal.

6°. If the point O' lies on the side BC, then the point lies on the$A$ circle $(A'B'C')$.

Hint. 1°. $4\overrightarrow{GO_9} = \overrightarrow{GH}$, $4\overrightarrow{O_9\alpha} = 2(\overrightarrow{HA_H} + \overrightarrow{OA_1}) = \overrightarrow{HA''} + \overrightarrow{AH} = \overrightarrow{HA''} + \overrightarrow{A''A'} = \overrightarrow{HA'}$.

2°. $4\overrightarrow{O_9\alpha} = \overrightarrow{HA'} = \overrightarrow{AA''}$.

3°. This is a consequence of item 2°.

4°. On the basis of item 1° we have $\overrightarrow{G\omega} = \frac{1}{4}\overrightarrow{GO'}$ and besides,

$$\overrightarrow{GO_9} = -\frac{1}{4}\overrightarrow{GH}, \quad \overrightarrow{O_9O_9'} = 2\overrightarrow{O_9\omega} = \frac{1}{2}\overrightarrow{HO'}$$

but we also have

$$\overrightarrow{OO_9} = \frac{1}{2}\overrightarrow{OH}.$$

5°. If P is an arbitrary point lying within $\triangle ABC$, then, denoting by A_2, B_2, C_2 the points of intersection of the straight lines PA', PB', PC' with the sides BC, CA, AB, we conclude that the lengths of the polygonal lines $PA_2A = PA'$, $PB_2B = PB'$, $PC_2C = PC'$ will be the shortest routes from point P to the vertices A, B, C with preliminary displacements to

the opposite sides. For these minimal distances to be equal, it is necessary and sufficient that $PA' = PB' = PC'$, that is, that the point P coincide with the point O'.

6° This is a consequence of item 5°.

9. Given three noncoplanar vectors

$$\overrightarrow{OA} = \mathbf{a}, \quad \overrightarrow{OB} = \mathbf{b}, \quad \overrightarrow{OC} = \mathbf{c}.$$

Let S be the centre of a sphere passing through the points O, A, B, C. Find the vector $\overrightarrow{OS} = \mathbf{x}$.

Answer. $\mathbf{x} = \dfrac{a^2[\mathbf{b}, \mathbf{c}] + b^2[\mathbf{c}, \mathbf{a}] + c^2[\mathbf{a}, \mathbf{b}]}{2(\mathbf{a}, \mathbf{b}, \mathbf{c})}$.

10. Given the vectors

$$\overrightarrow{OA} = \mathbf{a}, \quad \overrightarrow{OB} = \mathbf{b}, \quad \overrightarrow{OC} = \mathbf{c}.$$

The vectors **b** and **c** are noncollinear. Let H be the orthogonal projection of point A on the plane OBC. Find the vector $\overrightarrow{OH} = \mathbf{h}$.

Answer. $\mathbf{h} = \mathbf{a} - \dfrac{(\mathbf{a}, \mathbf{b}, \mathbf{c})}{[\mathbf{b}, \mathbf{c}]^2}[\mathbf{b}, \mathbf{c}]$.

11. Given four vectors

$$\overrightarrow{OA} = \mathbf{a}, \quad \overrightarrow{OB} = \mathbf{b}, \quad \overrightarrow{OC} = \mathbf{c}, \quad \overrightarrow{OD} = \mathbf{d}.$$

It is also given that the vectors **a**, **b**, **c** are noncoplanar and that the straight line OD intersects the plane ABC at some point M. Find the vector $\overrightarrow{OM} = \mathbf{m}$.

Answer. $\mathbf{m} = \dfrac{(\mathbf{a}, \mathbf{b}, \mathbf{c})}{(\mathbf{d}, \mathbf{b}, \mathbf{c}) + (\mathbf{d}, \mathbf{c}, \mathbf{a}) + (\mathbf{d}, \mathbf{a}, \mathbf{b})}\mathbf{d}$.

12. Given three vectors

$$\overrightarrow{OA} = \mathbf{a}, \quad \overrightarrow{OB} = \mathbf{b}, \quad \overrightarrow{OC} = \mathbf{c}.$$

The vectors **a** and **b** are noncollinear. Let H be the orthogonal projection of the point C on the plane OAB. Find the vector $\overrightarrow{CH} = \mathbf{x}$.

Answer. $\mathbf{x} = -\dfrac{(\mathbf{a}, \mathbf{b}, \mathbf{c})}{[\mathbf{a}, \mathbf{b}]^2}[\mathbf{a}, \mathbf{b}]$.

13. Find the vector **x** if three noncoplanar vectors **a**, **b**, **c** and their scalar products into **x** are known:

$$\mathbf{xa} = p, \quad \mathbf{xb} = q, \quad \mathbf{xc} = r.$$

Answer. $\mathbf{x} = \dfrac{p[\mathbf{b}, \mathbf{c}] + q[\mathbf{c}, \mathbf{a}] + r[\mathbf{a}, \mathbf{b}]}{(\mathbf{a}, \mathbf{b}, \mathbf{c})}$.

14. Find the vectors $\mathbf{x}$ and $\mathbf{y}$ if we know their sum $\mathbf{x}+\mathbf{y}=\mathbf{a}$, the scalar product $(\mathbf{x}, \mathbf{y}) = p$ and the vector product $[\mathbf{x}, \mathbf{y}] = \mathbf{b}$.

Answer. If $a^4 < 4(b^2 + pa^2)$, then there are no solutions. If $a^4 = 4(b^2 + pa^2)$, then there is one solution:

$$\mathbf{x} = \frac{\mathbf{a}}{2} + \frac{[\mathbf{a}, \mathbf{b}]}{a^2}, \quad \mathbf{y} = \frac{\mathbf{a}}{2} - \frac{[\mathbf{a}, \mathbf{b}]}{a^2}.$$

If $a^4 > 4(b^2 + pa^2)$, then there are two solutions:

$$\mathbf{x}_1 = \frac{a^2 + \sqrt{a^4 - 4(b^2+pa^2)}}{2a^2}\mathbf{a} + \frac{[\mathbf{a}, \mathbf{b}]}{a^2},$$

$$\mathbf{y}_1 = \frac{a^2 - \sqrt{a^4 - 4(b^2+pa^2)}}{2a^2}\mathbf{a} - \frac{[\mathbf{a}, \mathbf{b}]}{a^2},$$

$$\mathbf{x}_2 = \frac{a^2 - \sqrt{a^4 - 4(b^2+pa^2)}}{2a^2}\mathbf{a} - \frac{[\mathbf{a}, \mathbf{b}]}{a^2},$$

$$\mathbf{y}_2 = \frac{a^2 + \sqrt{a^4 - 4(b^2+pa^2)}}{2a^2}\mathbf{a} + \frac{[\mathbf{a}, \mathbf{b}]}{a^2}.$$

15. $ABCD$ is an arbitrary tetrahedron located in oriented space; P is an arbitrary point. The vectors $\overrightarrow{PA}, \overrightarrow{PB}, \overrightarrow{PC}, \overrightarrow{PD}$ are linearly dependent:

$$\alpha\overrightarrow{PA} + \beta\overrightarrow{PB} + \gamma\overrightarrow{PC} + \delta\overrightarrow{PD} = \mathbf{0}$$

(at least one of the numbers $\alpha, \beta, \gamma, \delta$ is different from zero). Prove that

$$\alpha : \beta : \gamma : \delta = (PBCD) : (APCD) : (ABPD) : (ABCP).$$

Hint. Project the zero vector $\alpha\overrightarrow{PA} + \beta\overrightarrow{PB} + \gamma\overrightarrow{PC} + \delta\overrightarrow{PD}$ on the straight line AB by planes parallel to the plane PCD. The points P, C, D are projected into the single point P_1 and we obtain $\alpha\overrightarrow{P_1A} + \beta\overrightarrow{P_1B} = \mathbf{0}$, and so on.

16. Given a trihedral angle $OABC$ such that among its plane angles BOC, COA, AOB there is not more than one angle equal to $\pi/2$. Prove vectorially that the three straight lines that pass through the vertex O, lie in the planes of the faces BOC, COA, AOB, and are perpendicular to the edges OA, OB, OC, respectively, lie in one plane.

Hint. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the direction vectors of the edges, then the direction vectors of the indicated straight lines are $[\mathbf{a}, [\mathbf{b}, \mathbf{c}]]$, $[\mathbf{b}, [\mathbf{c}, \mathbf{a}]]$, $[\mathbf{c}, [\mathbf{a}, \mathbf{b}]]$. Their sum is zero.

17. Prove vectorially that if all edges of the trihedral angle $OABC$, all plane angles of which are right angles, are cut by a plane that does not pass through its vertex O, then the point of intersection of the altitudes

of the triangle obtained in the section coincides with the projection of the point O on the plane of that triangle.

18. *ABCD* is an arbitrary tetrahedron: A', B', C', D' are the orthogonal projections of its vertices on the planes of the opposite faces. Given the radius vectors

$$\overrightarrow{DA} = \mathbf{r}_1, \quad \overrightarrow{DB} = \mathbf{r}_2, \quad \overrightarrow{DC} = \mathbf{r}_3.$$

1°. Find the radius vectors $\overrightarrow{DA'} = \mathbf{r}'_1$, $\overrightarrow{DB'} = \mathbf{r}'_2$, $\overrightarrow{DC'} = \mathbf{r}'_3$, $\overrightarrow{DD'} = \mathbf{r}'_4$ and prove that if the straight lines AA' and BB' lie in one plane, then $AB \perp CD$, and conversely.

2°. If it is also given that $AC = AD = BC = BD$, then prove that the straight lines AA' and BB' lie in one plane. Let H be the point of intersection of these straight lines, and let K be the point of intersection of the straight lines CC' and DD'. Prove that the plane AHB intersects the segment CD at its midpoint I and the plane CKD intersects the segment AB at its midpoint J. Prove that the point H is located on the straight line IJ.

19. Given, in a pyramid $OABC$, the length of the edge $OA = a$ and the plane angles $\angle BOC = \alpha$, $\angle COA = \beta$, $\angle AOB = \gamma$. A sphere (S) is tangent to the face BOC at the point O and passes through the point A. Find its radius x.

Solution. Consider the vector $\mathbf{x} = \overrightarrow{OS}$. Since the vector $\mathbf{x}$ is perpendicular to the plane BOC, it follows that $\mathbf{x} = \lambda[\mathbf{b}, \mathbf{c}]$, where $\mathbf{b} = \overrightarrow{OB}$, $\mathbf{c} = \overrightarrow{OC}$. From the equality $OS = SA$, we find $\mathbf{x}^2 = (\mathbf{a} - \mathbf{x})^2$, where $\mathbf{a} = \overrightarrow{OA}$, whence $\mathbf{xa} = a^2/2$. Substituting $\lambda[\mathbf{b}, \mathbf{c}]$ for $\mathbf{x}$ in this equation, we obtain $\lambda(\mathbf{a}, \mathbf{b}, \mathbf{c}) = a^2/2$, whence $\lambda = \dfrac{a^2}{2(\mathbf{a}, \mathbf{b}, \mathbf{c})}$ and, consequently, $\overrightarrow{OS} = \mathbf{x} = \dfrac{a^2[\mathbf{b}, \mathbf{c}]}{2(\mathbf{a}, \mathbf{b}, \mathbf{c})}$. The length x of the vector $\mathbf{x}$ is equal to the radius R (of the sphere S):

$$R = |\mathbf{x}| = x = \frac{a^2|[bc]|}{2|(\mathbf{a}, \mathbf{b}, \mathbf{c})|} = \frac{a^2 bc \sin\alpha}{2|(\mathbf{a}, \mathbf{b}, \mathbf{c})|}$$

$$= \frac{a^2 bc \sin\alpha}{2abc\sqrt{1 + 2\cos\alpha\cos\beta\cos\gamma - \cos^2\alpha - \cos^2\beta - \cos^2\gamma}}.$$

Thus,

$$R = \frac{a \sin\alpha}{2\sqrt{1 + 2\cos\alpha\cos\beta\cos\gamma - \cos^2\alpha - \cos^2\beta - \cos^2\gamma}}.$$

20. Given, in a tetrahedron $OABC$, the lengths of the edges $OA = a$, $OB = b$, $OC = c$ and the plane angles at the vertex O: $\angle BOC = \alpha$,

$\angle COA = \beta$, $\angle AOB = \gamma$. Find the radius x of the sphere (S) circumscribed about the given pyramid.

Solution. Let $\mathbf{x} = \overrightarrow{OS}$, $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, $\mathbf{c} = \overrightarrow{OC}$; then

$$\mathbf{x}^2 = (\mathbf{x} - \mathbf{a})^2 = (\mathbf{x} - \mathbf{b})^2 = (\mathbf{x} - \mathbf{c})^2$$

whence

$$\mathbf{xa} = a^2/2, \quad \mathbf{xb} = b^2/2, \quad \mathbf{xc} = c^2/2.$$

By the Gibbs formula,

$$\mathbf{x} = \mathbf{xa}\cdot\mathbf{a}^* + \mathbf{xb}\cdot\mathbf{b}^* + \mathbf{xc}\cdot\mathbf{c}^*,$$

where $\mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*$ is the reciprocal triple of the triple $\mathbf{a}, \mathbf{b}, \mathbf{c}$, we find

$$\mathbf{x} = \frac{a^2}{2}\mathbf{a}^* + \frac{b^2}{2}\mathbf{b}^* + \frac{c^2}{2}\mathbf{c}^* = \frac{a^2[\mathbf{a}, \mathbf{c}]}{2(\mathbf{a}, \mathbf{b}, \mathbf{c})} + \frac{b^2[\mathbf{c}, \mathbf{a}]}{2(\mathbf{a}, \mathbf{b}, \mathbf{c})} + \frac{c^2[\mathbf{a}, \mathbf{b}]}{2(\mathbf{a}, \mathbf{b}, \mathbf{c})}$$

and, hence,

$$x = \sqrt{\mathbf{x}^2} = \frac{\sqrt{\{a^2[\mathbf{b}, \mathbf{c}] + b^2[\mathbf{c}, \mathbf{a}] + c^2[\mathbf{a}, \mathbf{b}]\}^2}}{2|(\mathbf{a}, \mathbf{b}, \mathbf{c})|} = \frac{\sqrt{T}}{2|(\mathbf{a}, \mathbf{b}, \mathbf{c})|},$$

where

$$\begin{aligned}
T = {} & a^4[\mathbf{b}, \mathbf{c}]^2 + b^4[\mathbf{c}, \mathbf{a}]^2 + c^4[\mathbf{a}, \mathbf{b}]^2 \\
& + 2a^2b^2[\mathbf{b}, \mathbf{c}]\cdot[\mathbf{c}, \mathbf{a}] + 2b^2c^2[\mathbf{c}, \mathbf{a}]\cdot[\mathbf{a}, \mathbf{b}] + 2c^2a^2[\mathbf{a}, \mathbf{b}]\cdot[\mathbf{b}, \mathbf{c}] \\
= {} & a^4b^2c^2 \sin^2\alpha + b^4c^2a^2 \sin^2\beta + c^4a^2b^2 \sin^2\gamma \\
& + 2a^2b^2(bc \cos\alpha \cdot ac \cos\beta - c^2ab \cos\gamma) \\
& + 2b^2c^2(ca \cos\beta \cdot ba \cos\gamma - a^2bc \cos\alpha) \\
& + 2c^2a^2(ab \cos\gamma \cdot cb \cos\alpha - b^2ca \cos\beta) \\
= {} & a^4b^2c^2 \sin^2\alpha + b^4c^2a^2 \sin^2\beta + c^4a^2b^2 \sin^2\gamma \\
& + 2a^3b^3c^2(\cos\alpha \cos\beta - \cos\gamma) + 2b^3c^3a^2(\cos\beta \cos\gamma - \cos\alpha) \\
& + 2c^3a^3b^2(\cos\gamma \cos\alpha - \cos\beta) \\
= {} & a^2b^2c^2 \{a^2 \sin^2\alpha + b^2 \sin^2\beta + c^2 \sin^2\gamma \\
& + 2ab(\cos\alpha \cos\beta - \cos\gamma) + 2bc(\cos\beta \cos\gamma - \cos\alpha) \\
& + 2ca(\cos\gamma \cos\alpha - \cos\beta)\} = a^2b^2c^2M,
\end{aligned}$$

$$|(\mathbf{a}, \mathbf{b}, \mathbf{c})| = abc\sqrt{1 + 2\cos\alpha \cos\beta \cos\gamma - \cos^2\alpha - \cos^2\beta - \cos^2\gamma} = abc \cdot \Delta.$$

To summarize:

$$x = \frac{\sqrt{M}}{2\Delta}.$$

21. Given the plane angles $a = \angle BOC$, $b = \angle COA$, $c = \angle AOB$ of a trihedral angle $OABC$ and the interior dihedral angles A, B, C that

are respectively opposite the plane angles. In item 6° of example 1, Sec. 2, proof was given that

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}=\frac{\Delta}{\Delta^*},$$

where Δ is the volume of the tetrahedron constructed on the unit vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, that is,

$$\Delta=(1+2\cos a\cos b\cos c-\cos^2 a-\cos^2 b-\cos^2 c)^{1/2}$$

and

$$\Delta^*=(1-2\cos A\cos B\cos C-\cos^2 A-\cos^2 B-\cos^2 C)^{1/2}$$

Consider the following special cases:

1°. $a=A$, $b=B$, $c=C$. Prove that in this case at least one of the angles A, B, C is equal to $\pi/2$ (*Hint*: $\Delta^2-\Delta^{*2}=0$); furthermore, two edges are perpendicular to a third; two dihedral angles of the trihedral angle $OABC$ are equal to $\pi/2$, and the third one is arbitrary.

2°. $a=A$, $b=\pi-B$, $c=\pi-C$. The reasoning is the same as in item 1°.

3°. $a=A$, $b=B$, $c=\pi-C$. Prove that in this case $\cos a\cos b\cos c$ and $\cos A\cos B\cos C$ have opposite signs. The plane angle c is found from the relation

$$\cos c=\frac{\cos A\cos B}{1+\sin A\sin B}.$$

4°. $a=\pi-A$, $b=\pi-B$, $c=\pi-C$. Prove that the edges of the trihedral angle $OABC$ are pairwise perpendicular.

22. $ABCD$ is an arbitrary tetrahedron. Let

$$\mathbf{x}=[\overrightarrow{DB},\overrightarrow{DC}],\ \mathbf{y}=[\overrightarrow{DC},\overrightarrow{DA}],\ \mathbf{z}=[\overrightarrow{DA},\overrightarrow{DB}],\ \mathbf{t}=[\overrightarrow{AC},\overrightarrow{AB}].$$

1°. Prove that

$$\mathbf{x}+\mathbf{y}+\mathbf{z}+\mathbf{t}=\mathbf{0}.$$

Derive from this fact that with any tetrahedron $ABCD$ it is possible to associate three spatial quadrangles whose sides are perpendicular to the faces of the tetrahedron and the lengths of the sides of each of the spatial quadrangles are proportional to the areas of the faces of the tetrahedron.

Let h_a, h_b, h_c, h_d be the altitudes of the tetrahedron $ABCD$, let (AB), (AC), (AD), (BC), (BD), (CD) be the interior dihedral angles of the tetrahedron, which angles are adjacent to the edges AB, AC, AD, BC, BD, CD. Prove that

2°. $$\frac{1}{h_a}=\frac{\cos(CD)}{h_b}+\frac{\cos(DB)}{h_c}+\frac{\cos(BC)}{h_d}.$$

3°. $$\frac{1}{h_a^2}+\frac{1}{h_b^2}-2\frac{\cos(CD)}{h_ah_b}=\frac{1}{h_c^2}+\frac{1}{h_d^2}-2\frac{\cos(AB)}{h_ch_d}.$$

4°. $\frac{1}{2}\left(\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}+\frac{1}{h_d^2}\right)$

$$=\frac{\cos(AB)}{h_c h_d}+\frac{\cos(AC)}{h_b h_d}+\frac{\cos(AD)}{h_b h_c}+\frac{\cos(BC)}{h_a h_d}+\frac{\cos(BD)}{h_a h_c}+\frac{\cos(CD)}{h_a h_b}.$$

5°. $$\begin{vmatrix} -1 & \cos(CD) & \cos(BD) & \cos(BC) \\ \cos(CD) & -1 & \cos(AD) & \cos(AC) \\ \cos(BD) & \cos(AD) & -1 & \cos(AB) \\ \cos(BC) & \cos(AC) & \cos(AB) & -1 \end{vmatrix} = 0.$$

Hint. Form the scalar product of the equation $\mathbf{x}+\mathbf{y}+\mathbf{z}+\mathbf{t}=\mathbf{0}$ by $\mathbf{x}$; write down the equation $\mathbf{x}+\mathbf{y}+\mathbf{z}+\mathbf{t}=\mathbf{0}$ as $\mathbf{x}+\mathbf{y}=-\mathbf{z}-\mathbf{t}$ and square both sides; square the equation $\mathbf{x}+\mathbf{y}+\mathbf{z}+\mathbf{t}=\mathbf{0}$; eliminate h in the equations obtained in item 1°.

23. Prove that the six planes passing through the midpoints of the edges of the tetrahedron $ABCD$ and perpendicular to the opposite edges of the tetrahedron pass through the single point M (*Monge's point*). Prove that the Monge point is symmetric to the centre O of the sphere $(O)=(ABCD)$ with respect to the centroid G of the tetrahedron $ABCD$.

Hint. Let $\overrightarrow{OA}=\mathbf{r}_1$, $\overrightarrow{OB}=\mathbf{r}_2$, $\overrightarrow{OC}=\mathbf{r}_3$, $\overrightarrow{OD}=\mathbf{r}_4$. Then the equations of the planes indicated in the statement of the problem may be written thus:

$$\left(\mathbf{r}-\frac{\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4}{2}\right)(\mathbf{r}_i-\mathbf{r}_j)=0, \quad i\neq j \quad (i,j=1,2,3,4).$$

24. Given a tetrahedron $ABCD$ and a point M. Let the straight line passing through M parallel to the straight line AB intersect the faces CDA and CDB at the points P and Q. Prove that the sum of the scalar products $\overrightarrow{MP}\cdot\overrightarrow{MQ}$, the terms of which are made up as indicated for all six edges of the tetrahedron $ABCD$, is equal to the power of the point M with respect to the sphere $(ABCD)$.

25. Given a tetrahedron $ABCD$ and a point M. Prove that the sum of the powers of the vertex A with respect to the spheres with diameters MC, MB, MD is equal to the sum of the squares of the lengths of all edges of the tetrahedron $ABCD$ (and similarly for the vertices B, C, D).

26. Find the tetrahedron if we know the areas of its faces and also that the altitudes intersect in a single point (such a tetrahedron is said to be *orthocentric*).

Solution. Lagrange investigated this problem analytically (J. L. Lagrange Mem. Acad. Berlin, 1773, p. 160; OEuvres, t. III, p. 662) and formulated it slightly differently:

"Find a tetrahedron of greatest volume for which the areas of all its faces are given". From the formulas he obtained it follows that the desired tetrahedron is an orthocentric tetrahedron, as was also pointed out by

Serret (P. Serret.—J. de Liouville, 1862, p. 377), who proved this fact geometrically (without, however, going into the definition of such a tetrahedron). In connection with this problem, Lagrange obtained a fourth-degree equation and proved that it has at least one positive root; he computed the lengths of the edges of the tetrahedron as functions of that root, but did not consider the conditions under which a tetrahedron with such lengths of edges exists. Finally, an English mathematician Iyenger (Iyenger, *The Mathematics Student*, 1947, p. 104) solved the same problem by reducing it to a fourth-degree equation; he carried out a full investigation and proved that for such a tetrahedron to exist it is necessary and sufficient that the sum of the areas of any three faces be greater than the area of the fourth face. We now give the solution proposed by Marmion (A. Marmion, *Mathesis*, 1953, p. 69; problem No. 3253 proposed by V. Thébault).

As we know, the sum of four vectors **x**, **y**, **z**, **t** perpendicular to the edges of the tetrahedron $ABCD$ and directed outwards (the lengths of the vectors are equal to the areas of the faces of the tetrahedron) is equal to zero (see example 22). From this it follows that there exists a closed spatial quadrangle $A_1B_1C_1D_1$ whose sides are such that $\overrightarrow{A_1B_1} = \mathbf{x}$, $\overrightarrow{B_1C_1} = \mathbf{y}$, $\overrightarrow{C_1D_1} = \mathbf{z}$, $\overrightarrow{D_1A_1} = \mathbf{t}$ and, consequently, the sum of the areas of any three faces of the tetrahedron $ABCD$ exceeds the area of the fourth face.

Putting $\overrightarrow{DA} = \mathbf{a}$, $\overrightarrow{DB} = \mathbf{b}$, $\overrightarrow{DC} = \mathbf{c}$, we conclude that for the vectors **x**, **y**, **z** we can take the following vectors:

$$\mathbf{x} = \frac{1}{2}[\mathbf{b}, \mathbf{c}], \; \mathbf{y} = \frac{1}{2}[\mathbf{c}, \mathbf{a}], \; \mathbf{z} = \frac{1}{2}[\mathbf{a}, \mathbf{b}].$$

From the vectors **x**, **y**, **z**, **t** we can generate 6 spatial quadrangles of the type $A_1B_1C_1D_1$; all tetrahedrons $A_1B_1C_1D_1$ will have equal volume V_1:

$$V_1 = \frac{1}{6}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \frac{1}{48}([\mathbf{b}, \mathbf{c}], [\mathbf{c}, \mathbf{a}], [\mathbf{a}, \mathbf{b}])$$

$$= \frac{1}{48}(\mathbf{a}, \mathbf{b}, \mathbf{c})^2 = \frac{1}{48}(6V)^2 = \frac{3}{4}V^2.$$

From this it follows that the tetrahedrons $ABCD$ and $A_1B_1C_1D_1$ assume the maximum value simultaneously. On the other hand, the volume of any tetrahedron is equal to 2/3 of the product of the areas of two of its faces into the sine of the dihedral angle φ between them divided by the length of the edge of that dihedral angle, and so

$$V_1 = \frac{2}{3}\cdot\frac{\text{area}\,\triangle A_1B_1C_1\cdot\text{area}\,\triangle A_1D_1C_1}{A_1C_1}\sin\varphi,$$

where φ is the size of the dihedral angle with edge A_1C_1. Since the factor in front of $\sin\varphi$ does not depend on φ, it follows that V_1 is a maximum when $\varphi = \pi/2$. Similar reasoning carried out with respect to the other

faces $A_1B_1D_1$ and $C_1B_1D_1$ leads to the fact that the dihedral angle formed by these faces must also be a right angle. But since

$$A_1B_1 \perp DBC,\ B_1C_1 \perp ACD,\ C_1D_1 \perp BDA,\ D_1A_1 \perp CAB$$

it follows that

$$AB \perp A_1C_1D_1,\ BC \perp B_1D_1A_1,\ CD \perp C_1A_1B_1, DA \perp D_1B_1C_1$$

and, consequently, $AB \perp CD$, $BC \perp AD$, that is, the tetrahedron $ABCD$ of maximum volume for which the areas of its faces are given is orthocentric.

The coefficient of $\sin\varphi$ in the expression for V_1 depends solely on one variable $A_1C_1 = \lambda$. Applying the Heron formula for the areas of $\triangle A_1B_1C_1$ and $\triangle A_1D_1C_1$ and noting that $A_1B_1 = x$, $B_1C_1 = y$, $C_1D_1 = z$, $D_1A_1 = t$, we find that the square of the expression

$$\frac{\text{area } \triangle A_1B_1C_1 \cdot \text{area } \triangle A_1C_1D_1}{A_1C_1}$$

multiplied by 2^8 is equal to

$$F(\lambda^2) = \frac{[\lambda^2-(x+y)^2]\,[\lambda^2-(x-y)^2]\,[\lambda^2-(t+z)^2][\lambda^2-(t-z)^2]}{\lambda^2}.$$

Putting $\lambda^2 = \mu$, we find the following expression for the logarithmic derivative of the function $F(\mu)$:

$$\frac{F'(\mu)}{F(\mu)} = \frac{1}{\mu-(x+y)^2} + \frac{1}{\mu-(x-y)^2} + \frac{1}{\mu-(t+z)^2} + \frac{1}{\mu-(t-z)^2} - \frac{1}{\mu}.$$

Now let $(x-y)^2 \leqslant (t-z)^2 < (x+y)^2 \leqslant (t+z)^2$ since it must be true that

$$(t-z)^2 < A_1C_1^2 = \mu, \qquad (x+y)^2 > A_1C_1^2 = \mu$$

(only under this condition do triangles $A_1B_1C_1$ and $A_1D_1C_1$ exist with lengths of sides y, x, λ and t, z, λ). The derivative $F'(\mu)$ has only one root in the interval $((t-z)^2, (x+y)^2)$ and therefore there is only one value of λ for which the function $F(\lambda^2)$ attains an extremum, a maximum. Knowing λ, we can construct $\triangle A_1B_1C_1$ and $\triangle A_1C_1D_1$. Putting one against the other so that the dihedral angle $B_1(A_1C_1)D_1$ is equal to $\pi/2$, we construct the tetrahedron (and there is only one such tetrahedron to within isometry) that corresponds to the tetrahedron $ABCD$ with greatest volume. Furthermore, by virtue of the relations

$$AB \perp A_1C_1D_1,\ BC \perp B_1D_1A_1,\ CD \perp C_1A_1B_1,\ DA \perp D_1B_1C_1$$

and

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} = \mathbf{0},$$

we conclude that AB, BC, CD, DA must be proportional to the areas of the faces $A_1C_1D_1$, $A_1B_1D_1$, $B_1C_1A_1$, $C_1D_1B_1$ so that, denoting proportionality factor by k, we have

$$AB = k(A_1C_1D_1), \qquad BC = k(A_1B_1D_1),$$
$$CD = k(B_1C_1A_1), \qquad DA = k(C_1D_1B_1),$$

where, for instance, $(A_1B_1C_1)$ is the nonoriented area of face $A_1B_1C_1$, and on the basis of the foregoing, we have

$$V = \frac{3}{4}\, k^3V_1^2 = \frac{27}{64}\, k^3V^4,$$

whence we find the proportionality factor

$$\mathrm{k} = \frac{4}{3V} = \frac{2}{\sqrt{3V_1}}\,.$$

The lengths of the edges AB, BC, CD, DA have been determined and they must be perpendicular to the planes $A_1C_1D_1$, $A_1B_1D_1$, $B_1C_1A_1$, $C_1D_1B_1$ respectively. To summarize: up to isometric transformations (motions), there is only one tetrahedron that satisfies the statement of the problem.

CHAPTER II

ANALYTIC GEOMETRY

Sec. 1. Application of analytic geometry (solved problems)

Problem 1. The sides BC, CA, AB of a triangle ABC are divided by the points P, Q, R in the ratios

$$\frac{\overrightarrow{BP}}{\overrightarrow{PC}} = \lambda, \quad \frac{\overrightarrow{CQ}}{\overrightarrow{QA}} = \mu, \quad \frac{\overrightarrow{AR}}{\overrightarrow{RB}} = \nu.$$

Find $\dfrac{(PQR)}{(ABC)}$.

Solution. We introduce a general Cartesian coordinate system in the plane, putting $C = (0, 0)$, $A = (1, 0)$, $B = (0, 1)$. Then $P = \left(0, \dfrac{1}{1+\lambda}\right)$, $Q = \left(\dfrac{\mu}{1+\mu}, 0\right)$, $R = \left(\dfrac{1}{1+\nu}, \dfrac{\nu}{1+\nu}\right)$ and, consequently,

$$\frac{(PQR)}{(ABC)} = \begin{vmatrix} 0 & \dfrac{1}{1+\lambda} & 1 \\ \dfrac{\mu}{1+\mu} & 0 & 1 \\ \dfrac{1}{1+\nu} & \dfrac{\nu}{1+\nu} & 1 \end{vmatrix} = \frac{1+\lambda\mu\nu}{(1+\lambda)(1+\mu)(1+\nu)}.$$

Corollary. *For the points P, Q, R, lying on the sides BC, CA, AB of $\triangle ABC$, to be collinear, it is necessary and sufficient that the following equation hold:*

$$\frac{\overrightarrow{BP}}{\overrightarrow{PC}} \cdot \frac{\overrightarrow{CQ}}{\overrightarrow{QA}} \cdot \frac{\overrightarrow{AR}}{\overrightarrow{RB}} = -1 \quad \text{or} \quad \frac{\overrightarrow{BP}}{\overrightarrow{CP}} \cdot \frac{\overrightarrow{CQ}}{\overrightarrow{AQ}} \cdot \frac{\overrightarrow{AR}}{\overrightarrow{BR}} = 1$$

(Menelaus' theorem).

Problem 2. A triangle ABC is inscribed in a circle. Prove that the points P, Q, R of intersection of tangents to the circle at the points A, B, C are respectively collinear with the sides BC, CA, AB (this is *special case of the Brianchon theorem).*

Proof. Note that all the ratios

$$\lambda = \frac{\overrightarrow{BP}}{\overrightarrow{PC}}, \quad \mu = \frac{\overrightarrow{CQ}}{\overrightarrow{QA}}, \quad \nu = \frac{\overrightarrow{AR}}{\overrightarrow{RB}}$$

are negative. Furthermore,

$$\frac{BP}{PA} = \frac{\sin C}{\sin B}, \quad \frac{PC}{PA} = \frac{\sin B}{\sin C};$$

consequently,

$$\frac{\overrightarrow{BP}}{\overrightarrow{PC}} = \lambda = -\frac{\sin^2 C}{\sin^2 B} = -\frac{c^2}{b^2}.$$

Similarly, $\mu = -\frac{a^2}{c^2}$, $\nu = -\frac{b^2}{a^2}$ and so $\lambda\,\mu\,\nu = -1$ (see problem 1).

Problem 3. Three circles with centres A, B, C and radii R_1, R_2, R_3 are located so that each of them lies outside the other two. Let P, Q, R be the centres of exterior similitude of these circles taken in pairs. Prove that the points P, Q, R are collinear.

Proof. $\lambda = \frac{\overrightarrow{BP}}{\overrightarrow{PC}} = -\frac{R_2}{R_3}$, $\mu = \frac{\overrightarrow{CQ}}{\overrightarrow{QA}} = -\frac{R_3}{R_1}$, $\nu = \frac{\overrightarrow{AR}}{\overrightarrow{RB}} = -\frac{R_1}{R_2}$

whence $\lambda\,\mu\,\nu = -1$,

Problem 4. The straight lines AA_1, BB_1, CC_1 belong to one pencil (in particular, they pass through one point O). Prove that the points P, Q, R of intersection of the corresponding sides BC and B_1C_1, CA and C_1A_1, AB and A_1B_1 of the triangles ABC and $A_1B_1C_1$ are collinear *(Desargues' theorem).*

Solution. Consider $\triangle OBC$ and the transversal B_1PC_1 (the *transversal of a triangle* is any straight line lying in the plane of the triangle). On the basis of Menelaus' theorem we have

$$\frac{\overrightarrow{OB_1}}{\overrightarrow{B_1B}}\,\frac{\overrightarrow{BP}}{\overrightarrow{PC}}\,\frac{\overrightarrow{CC_1}}{\overrightarrow{C_1O}} = -1.$$

Similarly,

$$\frac{\overrightarrow{OC_1}}{\overrightarrow{C_1C}}\,\frac{\overrightarrow{CQ}}{\overrightarrow{QA}}\,\frac{\overrightarrow{AA_1}}{\overrightarrow{A_1O}} = -1,$$

$$\frac{\overrightarrow{OA_1}}{\overrightarrow{A_1A}}\,\frac{\overrightarrow{AR}}{\overrightarrow{RB}}\,\frac{\overrightarrow{BB_1}}{\overrightarrow{B_1O}} = -1.$$

Multiplying these equalities term by term, we get

$$\frac{\overrightarrow{BP}}{\overrightarrow{PC}}\frac{\overrightarrow{CQ}}{\overrightarrow{QA}}\frac{\overrightarrow{AR}}{\overrightarrow{RB}} = -1.$$

It is now left to the reader to prove Desargues' theorem for the case where AA_1, BB_1, CC_1 belong to one and the same improper pencil, that is to say, are parallel.

Problem 5. 1°. The sides $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$ of $\triangle ABC$ are divided in the ratios

$$\frac{\overrightarrow{BP}}{\overrightarrow{PC}} = \lambda \neq 0, \quad \frac{\overrightarrow{CQ}}{\overrightarrow{QA}} = \mu \neq 0, \quad \frac{\overrightarrow{AR}}{\overrightarrow{RB}} = \nu \neq 0.$$

Suppose the straight lines BQ and CR intersect in the point A_1, the straigh lines CR and AP in the point B_1, and the straight lines AP and BQ in the point C_1. Find the ratio $\dfrac{(A_1B_1C_1)}{(ABC)}$.

2°. Consider the special case $\lambda = \mu = \nu = 2$.

3°. Under what necessary and sufficient condition do the straight lines AP, BQ, CR belong to the same proper pencil?

4°. Under what necessary and sufficient condition do the straight lines AP, BQ, CR belong to the same improper pencil?

Solution. 1°. Introduce a general Cartesian coordinate system and set $C = (0, 0)$, $A = (1, 0)$, $B = (0, 1)$. Then

$$P = \left(0, \frac{1}{1+\lambda}\right), \quad Q = \left(\frac{\mu}{1+\mu}, 0\right), \quad R = \left(\frac{1}{1+\nu}, \frac{\nu}{1+\nu}\right).$$

The equations of the straight lines AP, BQ, CR are:

$$\begin{vmatrix} x & y & 1 \\ 1 & 0 & 1 \\ 0 & \dfrac{1}{1+\lambda} & 1 \end{vmatrix} = 0 \text{ or } x + (1+\lambda)y - 1 = 0 \text{ (straight line } B_1C_1 \text{ or } AP)$$

$$\begin{vmatrix} x & y & 1 \\ 0 & 1 & 1 \\ \dfrac{\mu}{1+\mu} & 0 & 1 \end{vmatrix} = 0 \text{ or } (1+\mu)\,x + \mu y - \mu = 0 \text{ (straight line } C_1A_1 \text{ or } BQ),$$

$$\begin{vmatrix} x & y & 1 \\ \dfrac{1}{1+\nu} & \dfrac{\nu}{1+\nu} & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0 \text{ or } \nu x - y = 0 \text{ (straight line } A_1B_1 \text{ or } CR)$$

whence

$$\frac{(A_1B_1C_1)}{(ABC)} = \frac{\begin{vmatrix} 1 & 1+\lambda & -1 \\ 1+\mu & \mu & -\mu \\ \nu & -1 & 0 \end{vmatrix}^2}{(1+\mu+\mu\nu)(1+\nu+\lambda\nu)(1+\lambda+\mu\lambda)}$$

$$= \frac{(\lambda\mu\nu - 1)^2}{(1+\mu+\mu\nu)(1+\nu+\lambda\nu)(1+\lambda+\mu\lambda)}.$$

Remark. By solving the systems of the straight lines B_1C_1, C_1A_1, A_1B_1, we could have found the vertices of the triangle $A_1B_1C_1$:

$$A_1 = \left(\frac{\mu}{1+\mu+\mu\nu}, \frac{\nu\mu}{1+\mu+\mu\nu}\right),$$

$$B_1 = \left(\frac{1}{1+\nu+\nu\lambda}, \frac{\nu}{1+\nu+\nu\lambda}\right),$$

$$C_1 = \left(\frac{\lambda\mu}{1+\lambda+\lambda\mu}, \frac{1}{1+\lambda+\lambda\mu}\right),$$

and then

$$\frac{(A_1B_1C_1)}{(ABC)} = \frac{\begin{vmatrix} \mu & \mu\nu & 1+\mu+\mu\nu \\ 1 & \nu & 1+\nu+\lambda\nu \\ \lambda\mu & 1 & 1+\lambda+\lambda\mu \end{vmatrix}}{(1+\lambda+\lambda\mu)(1+\mu+\mu\nu)(1+\nu+\nu\lambda)}.$$

By subtracting from the last column the sum of the first two and simplifying, we obtain the very same result.

2°. For $\lambda = \mu = \nu = 2$, we find $\dfrac{(A_1B_1C_1)}{(ABC)} = \dfrac{1}{7}$.

3°. The straight lines AP, BQ, CR belong to a single proper pencil if and only if $\lambda\mu\,\nu = 1$, $1+\lambda+\lambda\mu \neq 0$ (by multiplying both sides of the inequality $1+\lambda+\lambda\mu \neq 0$ by $\nu \neq 0$, we obtain $1+\nu+\nu\lambda \neq 0$; then, multiplying by $\mu \neq 0$, we obtain $1+\mu+\mu\nu \neq 0$).

4°. The straight lines AP, BQ, CR belong to the same improper pencil, if and only if $\lambda\mu\nu = 1$, $1+\lambda+\lambda\mu = 0$ (in this case, the lines AP, BQ CR are pairwise parallel, and there are no coincident lines).

Combining items 3° and 4°, we obtain a theorem: *if the sides BC, CA, AB of $\triangle\ ABC$ are divided in the ratios*

$$\frac{\overrightarrow{BP}}{\overrightarrow{PC}} = \lambda \neq 0, \quad \frac{\overrightarrow{CQ}}{\overrightarrow{QA}} = \mu \neq 0, \quad \frac{\overrightarrow{AR}}{\overrightarrow{RB}} = \nu \neq 0,$$

then the straight lines AP, BQ, CR belong to a single pencil (proper or improper) if and only if $\lambda\mu\nu = 1$ (Ceva's theorem).

Problem 6. Prove that if a circle inscribed in $\triangle ABC$ is tangent to the sides BC, CA, AB at the points, respectively, P, Q, R, then the straight lines AP, BQ, CR intersect in one point.

Solution.

$$\lambda = \frac{\overrightarrow{BP}}{\overrightarrow{PC}} = \frac{p-b}{p-c}, \quad \mu = \frac{\overrightarrow{CQ}}{\overrightarrow{QA}} = \frac{p-c}{p-a}, \quad \nu = \frac{\overrightarrow{AR}}{\overrightarrow{RB}} = \frac{p-a}{p-b}$$

($p = (a+b+c)/2$, a, b, c are the lengths of the sides of $\triangle ABC$), whence

$$\lambda\mu\nu = 1, \quad 1 + \lambda + \lambda\mu \neq 0.$$

Problem 7. Two distinct points A and B are fixed in a plane. Also fixed is a real number k. Find the locus of points M for each of which

$$MA^2 + MB^2 = k.$$

Solution. We introduce a rectangular Cartesian system of coordinates. Let

$$A = (x_1, y_1), \quad B = (x_2, y_2), \quad M = (x, y).$$

Using the formula for the distance between two points in a plane, we get

$$(x - x_1)^2 + (y - y_1)^2 + (x - x_2)^2 + (y - y_2)^2 = k$$

or

$$\left(x - \frac{x_1 + x_2}{2}\right)^2 + \left(y - \frac{y_1 + y_2}{2}\right)^2 = \frac{k}{2} - \frac{1}{4}[(x_2 - x_1)^2 + (y_2 - y_1)^2]$$

or

$$MO^2 = \frac{k}{2} - \frac{1}{4}AB^2,$$

where O is the midpoint of segment AB. From the last relation, it follows that if $k < AB^2/2$, then the specified locus is empty. If $k = AB^2/2$, then the specified locus consists of the single point O, the midpoint of AB. If $k > AB^2/2$, then the given locus of points M is a circle with centre O and radius $\sqrt{2k - AB^2}/2$.

Problem 8. Three points A, B, C are fixed in a plane. Also fixed is a real number k. Find the locus of points M for each of which

$$MA^2 + MB^2 + MC^2 = k.$$

Solution. Introduce in the plane of $\triangle ABC$ a rectangular Cartesian coordinate system. In this system, let

$$A = (x_1, y_1), \quad B = (x_2, y_2), \quad C = (x_3, y_3), \quad M = (x, y).$$

Applying the formula for the distance between two points, we obtain

$$(x-x_1)^2+(y-y_1)^2+(x-x_2)^2+(y-y_2)^2+(x-x_3)^2+(y-y_3)^2=k$$

or

$$3\left[\left(x-\frac{x_1+x_2+x_3}{3}\right)^2+\left(y-\frac{y_1+y_2+y_3}{3}\right)^2\right]$$
$$+\frac{1}{3}[(x_3-x_2)^2+(y_3-y_2)^2+(x_1-x_3)^2$$
$$+(y_1-y_3)^2+(x_2-x_1)^2+(y_2-y_1)^2]=k.$$

Since $\frac{x_1+x_2+x_3}{3}$, $\frac{y_1+y_2+y_3}{3}$ are the coordinates of G, the point of intersection of the medians of the triangle ABC, and

$$(x_3-x_2)^2+(y_3-y_2)^2=a^2,$$
$$(x_1-x_3)^2+(y_1-y_3)^2=b^2,$$
$$(x_2-x_1)^2+(y_2-y_1)^2=c^2,$$

where a, b, c are the lengths of the sides BC, CA, AB of the triangle, it follows that the last relation may be rewritten thus:

$$MG^2=\frac{1}{9}[3k-(a^2+b^2+c^2)].$$

From this it follows that if $k<(a^2+b^2+c^2)/3$, then the desired locus of points M is empty. If $k=(a^2+b^2+c^2)/3$ then the locus of points M consists of the single point G, which is the point of intersection of the medians of the triangle ABC. If $k>(a^2+b^2+c^2)/3$, then the locus is a circle with centre G and radius $\sqrt{3k-(a^2+b^2+c^2)}/3$.

Problem 9. Fixed in the plane are two distinct points C_1 and C_2 and also a real number k. Find the locus of points M for which

$$MC_1^2-MC_2^2=k.$$

Solution. Let us introduce a rectangular Cartesian coordinate system and take $\overrightarrow{C_1C_2}$ for the x-axis and the midpoint O of segment C_1C_2 for the coordinate origin. In that coordinate system, let

$$C_1=(-a,0),\quad C_2=(a,0),\quad M=(x,y).$$

Then the given relation takes the form

$$(x+a)^2+y^2-(x-a)^2-y^2=k \text{ or } x=\frac{k}{4a},$$

which is the equation of a straight line perpendicular to C_1C_2 (the x-axis).

Problem 10. The *radical axis* of two nonconcentric circles (C_1, r_1) and (C_2, r_2) is the locus of points M for each of which its powers with respect to the circles are equal. Set up the equation of the radical axis of the circles (C_1, r_1) and (C_2, r_2) assuming $\overrightarrow{C_1C_2}$ to be the x-axis and taking for the coordinate origin O the midpoint of the line segment C_1C_2.

Solution. In our coordinate system, $C_1=(-a, 0)$, $C_2=(a, 0)$, $M=(x, y)$. The relation

$$\sigma_1 = \sigma_2$$

or

$$MC_1^2 - r_1^2 = MC_2^2 - r_2^2$$

or

$$MC_1^2 - MC_2^2 = r_1^2 - r_2^2$$

takes the form

$$(x + a)^2 + y^2 - (x - a)^2 - y^2 = r_1^2 - r_2^2$$

or

$$x = \frac{r_1^2 - r_2^2}{4a}.$$

This is a straight line perpendicular to the straight line C_1C_2 (also see the preceding problem).

Remark. Consider the construction of a radical axis of two circles depending on their different mutual positions.

(1) The circles (C_1) and (C_2) intersect. Their radical axis is the straight line l that passes through the points P and Q (their points of intersection). Indeed, the powers of both points P and Q with respect to (C_1) and (C_2) are zero and hence the points P and Q belong to the radical axis of those circles; therefore the radical axis itself is the straight line PQ.

(2) The circles (C_1) and (C_2) are located so that each lies outside the other. Draw one of four straight lines tangent to the given circles; P_k, Q_k $(k = 1, 2, 3, 4)$ are points of contact; $P_k Q_k$ are segments of the tangents. Let the point M_k bisect the segment $P_k Q_k$. Then the point M_k lies on the radical axis of the given circles. Indeed, $M_k P_k^2 = M_k Q_k^2 = k$, where k is the power of the point M_k with respect to both circles. Thus, the midpoints of the four segments $P_k Q_k$ of the common tangents to the circles (C_1) and (C_2) bounded by the points of tangency P_k and Q_k lie on the radical axis of the circles. To construct the radical axis, it suffices to construct two of the points M_1, M_2, M_3, M_4 or even one (and then drop from it a perpendicular to the line of centres C_1C_2).

Remark. If the nonconcentric circles (C_1) and (C_2) do not intersect, their radical axis does not have any points in common with either of them. Indeed, suppose that the radical axis l of the circles (C_1) and (C_2) has a

common point A with circle (C_1). Then the power of point A with respect to the circle (C_1) is equal to zero, and since point A lies on the radical axis of (C_1) and (C_2), it follows that the power of A with respect to the circle (C_2) is also zero and therefore A also lies on the circle (C_2). This is a contradiction: the circles (C_1) and (C_2) have a common point A, which runs counter to the assumption.

It is left to the reader to prove a stronger statement: if each of the circles (C_1), (C_2) lies outside the other, then (C_1) and (C_2) lie on different sides of their radical axis, and if one of the circles (C_1), (C_2) lies inside the other (see below), then (C_1) and (C_2) lie on one side of their radical axis [in all instances we are dealing with two nonconcentric circles (C_1) and (C_2)].

(3) The circle (C_1) is a zero circle (point C_1) and lies outside the circle (C_2). Construct the tangent lines C_1T_1 and C_2T_2 drawn from the point C_1 to the circle (C_2) (T_1 and T_2 are the points of tangency). Suppose M_1 and M_2 are the midpoints of segments C_1T_1 and C_2T_2; the straight line M_1M_2 is the radical axis of the circles (C_2) and (C_1). One could construct a single tangent line C_1T_1, the radical axis is a straight line passing through the midpoint M_1 of segment C_1T_1 perpendicularly to the straight line C_1C_2.

(4) The circles (C_1) and (C_2) are zero circles. Their radical axis is the perpendicular bisector of the line segment C_1C_2.

(5) The circles (C_1) and (C_2) are nonzero and are tangent to one another externally or internally. Their radical axis is the tangent line l at their common point T. Indeed, the powers of any point M lying on the tangent line l with respect to (C_1) and (C_2) are equal to MT^2, which means they are equal.

(6) The zero circle (C_1) "lies" on the nonzero circle (C_2). The radical axis of these circles is the tangent line to the circle (C_2) at the point C_1. Indeed, the point C_1 belongs to the radical axis since its powers with respect to (C_1) and (C_2) are zero, and, besides, the radical axis is perpendicular to the straight line C_1C_2.

(7) The circle (C_1) is a zero circle, the circle (C_2) is nonzero and the point C_1 lies inside (C_2); also the circles are nonconcentric ($C_1 \neq C_2$). Let us construct some circle (C) tangent to the straight line C_1C_2 at the point C_1 and such that it intersects (C_2) at the points A and B. Then the straight line AB intersects the straight line C_1C_2 at the point P lying on the radical axis of the circles (C_1) and (C_2). Indeed, the power of the point P with respect to the circle (C_2) is equal to $PA \cdot PB$. But this product is equal to PC_1^2, that is, to the power of the point P with respect to the zero circle (C_1). The radical axis of the circles (C_1) and (C_2) is a straight line passing through P and perpendicular to the straight line C_1C_2.

(8) The nonconcentric circles (C_1) and (C_2) are nonzero and the circle (C_1) lies inside the circle (C_2). Let us construct some circle (C) which intersects (C_1) at the points A and B, and the circle (C_2) at the points A' and B'. Let P be the point of intersection of the straight lines AB and $A'B'$. Then P belongs to the radical axis of the circles (C_1) and (C_2), since its powers

$\sigma = (PA) \cdot (PB)$ and $\sigma' = (PA') \cdot (PB')$ with respect to the circles (C_1) and (C_2) are equal: $(PA) \cdot (PB) = (PA') \cdot (PB')$; this is the power of the point P with respect to the circle (C). The radical axis l of the circles (C_1) and (C_2) is a straight line passing through the point P perpendicularly to the straight line C_1C_2.

Problem 11. Fixed in a plane are two distinct points A and B and a positive number k not equal to unity.

1°. Prove that the locus of points M for which the following equation holds,

$$\frac{MA}{MB} = k,$$

is a circle (C_k)*, the centre C_k of which lies on the straight line AB. Introduce a rectangular Cartesian coordinate system taking for the origin the midpoint O of AB, and for the x-axis the straight line AB. On the x-axis find the coordinate x_k of the centre C_k of the circle (C_k) if we know the coordinates $-a$ and a of the points A and B (we assume $a > 0$, that is, the positive direction of the coordinate axis is chosen from A to B). Also find the radius R_k of the circle (C_k).

2°. Prove that the centre C_k of the circle (C_k) is an external point of the line segment AB.

3°. Prove that for any k $(0 < k \neq 1)$ the circle (C_k) does not have common points with the perpendicular bisector (or midperpendicular) s of AB: for $0 < k < 1$, the circle (C_k) and the point A lie to one side of the midperpendicular s of AB [and, consequently, B and the circle (C_k) lie on different sides of s], and for $k > 1$, the point A and the circle (C_k) lie on different sides of s [and, hence, B and the circle (C_k) lie on the same side of s].

4°. Prove that if $0 < k < 1$, then point A lies inside the circle (C_k) and point B outside it. If $k > 1$, then, conversely, point A lies outside (C_k) and B lies inside (C_k).

5°. Prove that the circles (C_{k_1}) and (C_{k_2}) $(0 < k_1 \neq 1,\ 0 < k_2 \neq 1)$ are symmetric with respect to the midperpendicular s of segment AB if and only if $k_1k_2 = 1$.

6°. Let P_k and Q_k be points of intersection of the circle (C_k) with the straight line AB. Prove that the ordered quadruplet of points A, B, P_k, Q_k is *harmonic*, that is,

$$\frac{\overrightarrow{AP_k}}{\overrightarrow{P_kB}} : \frac{\overrightarrow{AQ_k}}{\overrightarrow{Q_kB}} = -1.$$

* A circle thus defined is called a *circle of Apollonius*. The points A and B are termed *limit points*, or *Poncelet points* of the circles (C_k) (if k varies from 0 to $+\infty$, with the exception of $k = 1$, then we obtain a family of circles (C_k); the case of $k = 1$ is clearly associated with the straight line: the midperpendicular of points A and B).

7°. Prove that if S is an arbitrary point of the midperpendicular s of AB, and T is the point of contact of any tangent (any one of two) drawn from point S to the circle (C_k), then

$$ST = SA = SB.$$

In other words, the midperpendicular s of AB is the radical axis of any one of the circles (C_k) $(0 < k \neq 1)$, including the zero circles A and B.

8°. Prove that any one of the circles (S) passing through points A and B intersects all circles (C_k) $(0 < k \neq 1)$ orthogonally, that is, the tangents to the circles (S) and (C_k) at any one of two points of their intersection are mutually perpendicular.

The set (C_k) of all circles of Apollonius given by

$$\frac{MA}{MB} = k,$$

where k takes on all positive values (this set includes the midperpendicular of AB as well) is termed a *hyperbolic pencil of circles*. Points A and B are called the *limit points* of the pencil, or *Poncelet points*.

The set of all circles passing through points A and B (this set includes the straight line AB as well) is called an *elliptic pencil of circles* which is conjugate to the hyperbolic pencil. The points A and B are termed *base points* of the elliptic pencil.

Remark. A hyperbolic pencil of circles may be defined in a variety of ways (all the definitions given below are equivalent).

(1) A hyperbolic pencil Γ of circles is the image of a family of all concentric circles (O) under inversion. Here, the pencil of straight lines with centre O is transformed into an elliptic pencil E of circles which is conjugate to the hyperbolic pencil Γ.

(2) A hyperbolic pencil Γ of circles is a set of all circles with a common radical axis (that is, a set such that the radical axes of any two circles coinside); in this case the radical axis does not intersect the circles.

(3) A hyperbolic pencil Γ of circles is a set of all circles (PQ) with diameter PQ, where the points P and Q are harmonic conjugates of the limit points A and B (any two distinct points may be taken for A and B). The set of all circles passing through A and B then form an elliptic pencil E of circles conjugate to the hyperbolic pencil Γ.

(4) A hyperbolic pencil Γ of circles is a stereographic projection (see Chapter IV) from an arbitrary points of the sphere of the set of all circles of the sphere, the planes of which are perpendicular to any diameter NS of the sphere. In this case, the projections A and B of points N and S are limit points of the pencil Γ, and the stereographic projections of the set of great circles of the sphere that pass through the points N and S form an elliptic pencil E of circles conjugate to the pencil Γ.

(5) A hyperbolic pencil Γ of circles with limit points $A(-a, 0)$ and $B(a, 0)$ is a family of circles given by the equation

$$\lambda f + \mu\varphi = 0,$$

where

$$f \equiv x^2 + y^2 - 2px + a^2 = 0,$$
$$\varphi \equiv x^2 + y^2 - 2qx + a^2 = 0, \quad p \neq q$$

are any two circles of the family (at least one of the numbers λ and μ is nonzero).

(6) A hyperbolic pencil Γ of circles is a set of circles specified by the equation

$$x^2 + y^2 - 2px + 1 = 0,$$

where p is a parameter that takes on all values that do not exceed 1 in absolute value. Then the conjugate elliptic pencil is given by the equation

$$x^2 + y^2 - 2qx - 1 = 0,$$

where the parameter q takes on all real values; the fundamental points of the pencils Γ and E are $(\pm 1, 0)$.

There are other definitions as well.

9°. Let $0 < k_1 < k_2 < 1$. Prove that the circle (C_{k_1}) is inside the circle (C_{k_2}). But if $1 < k_1 < k_2$, then the circle (C_{k_2}) is inside the circle (C_{k_1}).

10°. Prove the validity of the following method of constructing a circle of Apollonius: given Poncelet points A and B and one of the points M_0 of the circle of Apollonius not lying either on the straight line AB or on the midperpendicular of the segment AB; at the point M_0 draw a tangent to the circle (ABM_0). If C is the point of intersection of that tangent with AB, then the circle of Apollonius passing through M_0 is a circle with centre C and radius CM_0. Consider the case where the point M_0 lies on the straight line AB but is distinct from the points A, B and the midpoint O of AB.

11°. Find the locus of points M for each of which

11°_1. $\dfrac{MA}{MB} > k$, where $0 < k \neq 1$,

11°_2. $\dfrac{MA}{MB} < k$, where $0 < k \neq 1$.

12°. Prove that two nonconcentric circles (C') and (C'') without common points define uniquely a hyperbolic pencil of circles to which they belong. How are the limit points of the pencil constructed? Consider the case where one or both circles are zero circles.

13°. Fixed in the plane are two distinct points A and B. Given a straight line l. On l find points M such that the ratio

$$\frac{MA}{MB}$$

assumes maximum and minimum values. Investigate the question depending on the position of the straight line l relative to the points A and B.

14°. Given: a plane π and a straight line l intersecting the plane in the point A. Given an angle φ that the straight line l forms with the plane π

(if l is not perpendicular to the plane π, then φ is an acute angle between l and its projection l' on the plane π; if l is perpendicular to the plane π, $\varphi = 90°$). A point B different from A is fixed on l.

Let PQ be an arbitrary line segment lying in space. Draw through points P and Q straight lines collinear with l and let P' and Q' be the points of intersection of these lines with the π-plane. The line segment $P'Q'$ is called the parallel projection of PQ on the π-plane in the direction of l. The ratio

$$k = \frac{P'Q'}{PQ}$$

is called the *deformation ratio* of segment PQ under its parallel projection on the π-plane in the direction of the straight line l (we will simply write deformation ratio because in this problem we consider the parallel projection only on the π-plane in the direction of the straight line l).

Let there be a fixed positive number k. In the π-plane, find the locus of points M each of which has the following property: the deformation ratio of any nonzero segment lying on the straight line BM is equal to k.

Solution. 1°. $A = (-a, 0)$, $B = (a, 0)$, $M = (x, y)$. Employing the formula for the distance between two points, we obtain

$$\frac{\sqrt{(x+a)^2+y^2}}{\sqrt{(x-a)^2+y^2}} = k$$

or

$$\left(x - a\frac{k^2+1}{k^2-1}\right)^2 + y^2 = \frac{4a^2k^2}{(k^2-1)^2},$$

this is the equation of a circle with centre at the point $C_k(x_k, 0)$, where

$$x_k = a\frac{k^2+1}{k^2-1},$$

and with radius

$$R_k = \frac{2ak}{|k^2-1|}.$$

2°. From the equation

$$x_k = a\frac{k^2+1}{k^2-1}$$

it follows that $|x_k| > a$.

3°. If the circle (C_k) had at least one point M_0 in common with the mid-perpendicular of AB, then for that point we would have

$$1 = \frac{M_0A}{M_0B} = k$$

but this contradicts the condition $k \neq 1$. If $0<k<1$, then $x_k = a\dfrac{k^2+1}{k^2-1} < 0$; hence, the points C_k and A lie on one side of the midperpendicular of AB and thus the point A and (C_k) lie on one side of the midperpendicular. If $k > 1$, then $x_k > 0$; hence, the points B and C_k lie on one side of the midperpendicular of segment AB, and since the circle (C_k) does not have any points in common with that midperpendicular, it follows that the point B and (C_k) lie on one side of the midperpendicular of AB.

4°. The power of the point A with respect to the circle (C_k) is equal to

$$\sigma_A = \left(-a - a\frac{k^2+1}{k^2-1}\right)^2 - \frac{4a^2k^2}{(k^2-1)^2} = \frac{4a^2k^2}{k^2-1}.$$

If $0 < k < 1$, then $\sigma_A < 0$, that is, point A lies inside (C_k) and, hence, point B lies outside (C_k) since the points A and B lie on different sides of the midperpendicular s of segment AB, and the point A and (C_k) lie (for $0 < k < 1$) on one side of the midperpendicular of AB; if $k > 1$, then $\sigma_A > 0$ and A lies outside (C_k).

The power σ_B of point B with respect to (C_k) is

$$\sigma_B = \left(a - a\frac{k^2+1}{k^2-1}\right)^2 - \frac{4a^2k^2}{(k^2-1)^2} = \frac{4a^2}{1-k^2}.$$

If $0 < k < 1$, then $\sigma_B > 0$, and if $k > 1$, then $\sigma_B < 0$.

5°. Prove that if $0 < k_1 < k_2 < 1$, then $R_{k_1} < R_{k_2}$, and if $1 < k_1 < k_2$, then $R_{k_1} > R_{k_2}$.

Indeed, if $0 < k_1 < k_2 < 1$, then

$$R_{k_1} - R_{k_2} = \frac{2ak_1}{1-k_1^2} - \frac{2ak_2}{1-k_2^2} = 2a\frac{(k_1-k_2)(1+k_1k_2)}{(1-k_1^2)(1-k_2^2)} < 0,$$

and if $1 < k_1 < k_2$, then

$$R_{k_1} - R_{k_2} = \frac{2ak_1}{k_1^2-1} - \frac{2ak_2}{k_2^2-1} = 2a\frac{(k_2-k_1)(1+k_1k_2)}{(k_1^2-1)(k_2^2-1)} > 0.$$

Thus, the circles (C_{k_1}) and (C_{k_2}) may be symmetric with respect to the midperpendicular s of segment AB in one of the following cases:

$$0 < k_1 < 1 < k_2, \qquad 0 < k_2 < 1 < k_1.$$

Both cases are investigated in the same way. For example, let $0 < k_1 < 1$, $k_2 > 1$; then

$$R_{k_1} = \frac{2ak_1}{1-k_1^2}, \qquad R_{k_2} = \frac{2ak_2}{k_2^2-1}.$$

A necessary condition for the circles (C_{k_1}) and (C_{k_2}) (in the case of $0 < k_1 < 1 < k_2$) to be symmetric with respect to the midperpendicular s of segment AB is that their radii be equal:

$$R_{k_1} = R_{k_2},$$

$$\frac{2ak_1}{1-k_1^2} = \frac{2ak_2}{k_2^2-1},$$

$$k_1 k_2^2 - k_1 - k_2 + k_1^2 k_2 = 0,$$

$$(k_1 + k_2)(k_1 k_2 - 1) = 0,$$

and since $k_1 + k_2 > 0$, it follows that $k_1k_2 - 1 = 0$, whence $k_1k_2 = 1$. This condition is also sufficient for the circles (C_{k_1}) and (C_{k_2}) to be symmetric with respect to the midperpendicular s of segment AB. Indeed, assuming that $k_1k_2 = 1$, we have

$$x_{k_1} = 2a\frac{k_1^2+1}{k_1^2-1} = 2a\frac{\dfrac{1}{k_2^2}+1}{\dfrac{1}{k_2^2}-1} = 2a\frac{1+k_2^2}{1-k_2^2} = -x_{k_2}.$$

Consequently the points C_{k_1} and C_{k_2} are symmetric with respect to the point O or, what is the same thing, with respect to the midperpendicular s of segment AB. But from the condition $k_1k_2 = 1$ it follows that $R_{k_1} = R_{k_2}$, which means the circles (C_{k_1}) and (C_{k_2}) themselves are symmetric with respect to the straight line s.

Remark. Everything stated in this item also follows from the fact that the given equation is equivalent to the following equation:

$$\frac{MB}{MA} = \frac{1}{k}.$$

6°. One of the points A, B lies inside the circle (C_k) and other outside the circle; to put it differently, one of the points A, B is an interior point of the diameter P_kQ_k of the circle (P_kQ_k) and the other is an exterior point. From this it follows that one of the points P_k, Q_k is an interior point of segment AB and the other is an exterior point. Therefore the numbers

$$\frac{\overrightarrow{AP_k}}{\overrightarrow{P_kB}}, \quad \frac{\overrightarrow{AQ_k}}{\overrightarrow{Q_kB}}$$

have different signs. But the absolute values of these ratios are equal to k, since the points P_k and Q_k lie on the circle (C_k) and, hence, belong to the given locus. Thus,

$$\frac{\overrightarrow{AP_k}}{\overrightarrow{P_kB}} : \frac{\overrightarrow{AQ_k}}{\overrightarrow{Q_kB}} = -1.$$

To summarize: the points P_k and Q_k are harmonic conjugates of the points A and B.

Conversely, if the points P_k and Q_k are harmonic conjugates of the pair A, B, then

$$\frac{AP_k}{P_kB}=\frac{AQ_k}{Q_kB}\ (=k),$$

that is, the points P_k and Q_k belong to the circle (C_k) of the hyperbolic pencil of circles with limit points A and B (for the indicated value of k). From this it follows that the circle constructed on the segment P_kQ_k as diameter is the circle (C_k).

The proposition just proven that A, B, P_k, Q_k is a harmonic set of four points permits the following elegant construction of circles of Apollonius: through points A and B draw two straight lines s_1 and s_2 perpendicular to the straight line AB (Fig. 2) and on them fix points S_1 and S_2 that are distinct from points A and B respectively. On the straight line s_2 lay off, on different sides of the straight line AB, congruent line segments BP'_k and BQ'_k. The straight lines $S_1P'_k$ and $S_1Q'_k$ intersect AB at the points P_k and Q_k. Then the straight lines

$$S_1A,\ S_1B,\ S_1P'_k,\ S_1Q'_k$$

form a harmonic set of four points. The circle with diameter P_kQ_k is the circle of a hyperbolic pencil with limit points A and B.

Figure 3 gives yet another method for constructing a circle of Apollonius: let l be a tangent to the circle Ω at the point S; N is a point on the circle Ω diametrically opposite to point S.

Construct a family of chords $P'_kQ'_k$ of the circle Ω parallel to its diameter SN. Let $A'B'$ be the diameter of the circle Ω that is perpendicular to the diameter SN, and let A and B be the projections of the points A' and B' on the straight line l from the point N. Denote by P_k and Q_k the projections of points P'_k and Q'_k on the straight line l from point N. The set of four points

$$A,\ B,\ P_k,\ Q_k$$

is a harmonic set since the quadruplet of straight lines

$$NA',\ NB',\ NP'_k,\ NQ'_k$$

is harmonic ($\angle A'NB'=\pi/2$ and NB' and NA' are bisectors of $\angle P'_kNQ'_k$). The circle with diameter P_kQ_k is a circle of Apollonius with limit points A and B.

This method of constructing circles of Apollonius is connected with the so-called stereographic projection of a sphere on a plane (this projection is used in the preparation of geographical maps).

7°. Let $S(0, b)$ be an arbitrary point of the midperpendicular s of segment AB. Since all points of the midperpendicular s of AB are external points of the circle (C_k), it follows that from the point S it is possible to

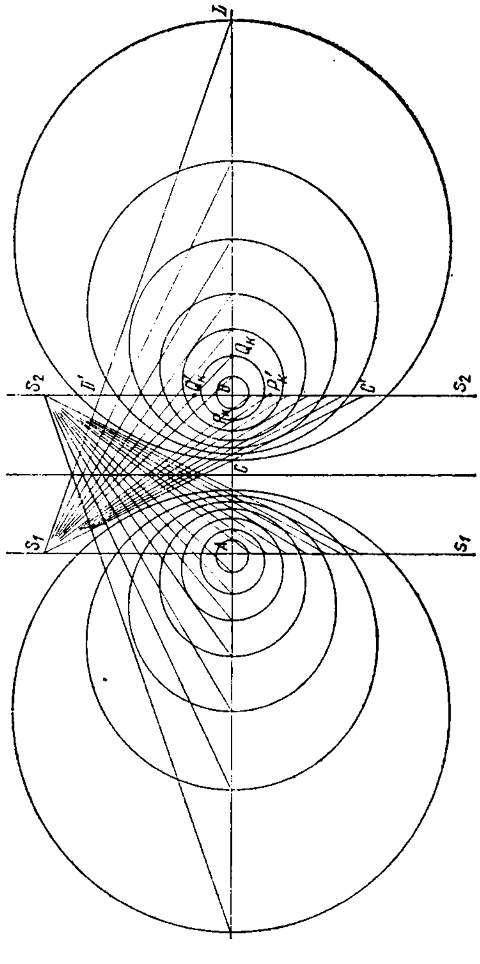

Fig. 2

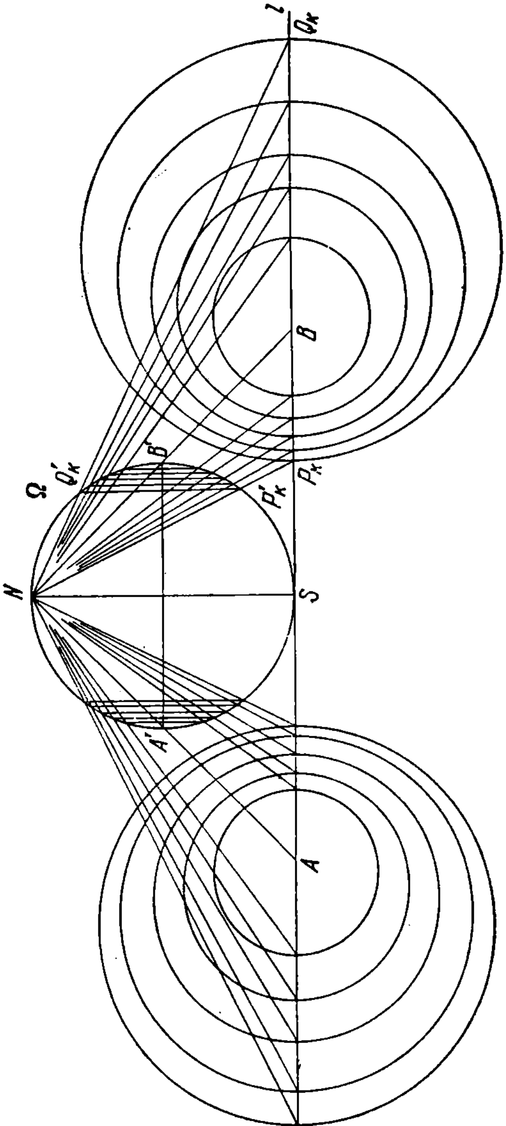

Fig. 3

draw two tangents to the circle (C_k). Let ST be one of them and let T be the point of tangency.

The power of point S with respect to the circle (C_k) is equal to ST^2:

$$\sigma = ST^2 = \left(a\,\frac{k^2+1}{k^2-1}\right)^2 + b^2 - \frac{4a^2k^2}{(k^2-1)^2} = a^2 + b^2$$

and, consequently,

$$ST = \sqrt{a^2+b^2} = \sqrt{OA^2 + OS^2} = SA = SB.$$

We see that the length of segment ST is independent of k, and, hence, all circles (C_k) and the zero circles A and B have a common radical axis, the midperpendicular s of AB.

8°. From what was proved in item 7° it follows that the circle whose centre is any point S of the midperpendicular s of AB and whose radius is equal to $SA = SB$ orthogonally intersects the circle (C_k), since $SA = SB = ST$, where T is the point of contact of the tangent (either one of the two) to (C_k) drawn from point S. Any circle passing through the points A and B has its centre on the midperpendicular of segment AB and its radius is equal to $SA = SB$.

9°. Let $0 < k_1 < k_2 < 1$. The circles (C_{k_1}) and (C_{k_2}) do not have any points in common for $k_1 \neq k_2$ because if they did have a common point M_0, then the following relations would hold true:

$$k_1 = \frac{M_0A}{M_0B} = k_2,$$

but this is not true ($k_1 \neq k_2$). In the case of $0 < k_1 < k_2 < 1$, point A lies inside both (C_{k_1}) and (C_{k_2}), but since these circles do not have any points in common, one of them lies inside the other. But if $0 < k_1 < k_2 < 1$ we have $R_{k_1} < R_{k_2}$ (see item 5°); this means the circle (C_{k_1}) lies inside (C_{k_2}).

But if $1 < k_1 < k_2$, then the point B lies inside (C_{k_1}) and (C_{k_2}) (see item 4°), the circles (C_{k_1}) and (C_{k_2}) do not have any points in common and $R_{k_1} > R_{k_2}$ (see item 5°); which means (C_{k_2}) lies inside (C_{k_1}).

10°. (1) Construct a circle (ABM_0). By what was proved in item 8°, the circle (C_k) of Apollonius must intersect (at point M_0) (ABM_0) orthogonally. Therefore the radius of (C_k) will lie on the tangent to (ABM_0) at the point M_0. On the other hand, the centre C_k of the circle (C_k) lies on line AB and therefore it is the point of intersection of the straight line AB with the tangent to (ABM_0) at the point M_0. The value of k that corresponds to the circle (C_k) is:

$$k = \frac{M_0A}{M_0B}$$

because the point M_0 lies on (C_k).

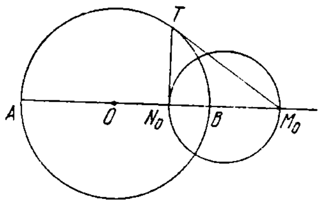

Fig. 4

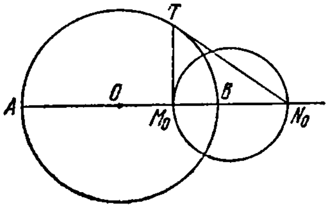

Fig. 5

(2) If point M_0 lies on the straight line AB but differs from the points A, B, O, then it is one of the points P_k, Q_k of intersection of (C_k) with that straight line. The other point N_0 of intersection of (C_k) with AB is found from the relation

$$(ABM_0 N_0) = -1,$$

whence

$$(OM_0)\cdot(ON_0) = a^2,$$

where O is the midpoint of AB; the point N_0 is the image of point M_0 under inversion with respect to the circle constructed on AB as a diameter.

The construction of point N_0 is given in Figs. 4 and 5; the point N_0 belongs to the circle of Apollonius passing through point M_0 because from the relation

$$(OM_0)\cdot(ON_0) = a^2$$

it follows that

$$(ABM_0 N_0) = -1,$$

$$\frac{AM_0}{BM_0} = \frac{AN_0}{BN_0} (= k).$$

The construction of point N_0, which is the harmonic conjugate of point M_0 with respect to A, B may be carried out in a great variety of ways (without recourse to inversion).

If a straight line parallel to AB is given in a plane, then that construction may be performed even with a straightedge alone.

11°. The inequality

$$\frac{MA}{MB} > k$$

or

$$MA^2 - k^2\, MB^2 > 0$$

is equivalent to the following:

$$(1 - k^2)\, x^2 + (1 - k^2)\, y^2 + 2a(1 + k^2)\, x + a^2\,(1 - k^2) > 0.$$

If $0 < k < 1$, then by dividing the left member of this inequality by $1 - k^2$ and simplifying we obtain

$$\left(x - a\frac{k^2+1}{k^2-1}\right) + y^2 - \frac{4a^2k^2}{(k^2-1)^2} > 0.$$

This inequality is satisfied by the coordinates of all points (x, y) lying *outside* the circle (C_k).

If $k > 1$, we get

$$\left(x - a\frac{k^2+1}{k^2-1}\right)^2 + y^2 - \frac{4a^2k^2}{(k^2-1)^2} < 0.$$

This inequality is satisfied by the coordinates of all points lying *within* (C_k).

In similar fashion, proof is given that for $0 < k < 1$ the inequality

$$\frac{MA}{MB} < k$$

is satisfied by all points M lying *inside* (C_k), and for $k > 1$, by all points lying *outside* (C_k)

12°. Let s be the radical axis of the circles (C') and (C'') and let O be the point at which the radical axis intersects the straight line $C'C''$ of centres of the given circles. Denote by P'_k, Q'_k, P''_k, Q''_k respectively the points of intersection of the circles (C') and (C'') with the straight line $C'C''$. Since the point O lies on the radical axis of (C') and (C''), it follows that its powers with respect to these circles are equal:

$$\sigma = (OP'_k)\cdot(OQ'_k) = (OP''_k)\cdot(OQ''_k)$$

(these powers are positive since the radical axis does not intersect the given circles and, hence, point O lies outside the circles).

Let us now construct on the straight line $C'C''$ the points A and B distant $\sqrt{\sigma}$ from O. Then

$$OA^2 = OB^2 = (OP'_k)\cdot(OQ'_k) = (OP''_k)\cdot(OQ''_k) = \sigma.$$

From these relations it follows that

$$(ABP'_k\,Q'_k) = -1, \qquad (ABP''_k\,Q''_k) = -1,$$

that is, (C') and (C'') are circles of Apollonius for values k equal to

$$k' = \frac{P'_kA}{P'_kB} = \frac{Q'_kA}{Q'_kB} \qquad k'' = \frac{P''_kA}{P''_kB} = \frac{Q''_kA}{Q''_kB}.$$

13°. Construct the midperpendicular s of segment AB. Suppose that a straight line l intersects this midperpendicular but does not pass through the points A and B [case (1) see Fig. 6]. Let S be the point of intersection of straight lines l and s. Now construct a circle with centre S and radius SA; it intersects the straight line l in two points M_1 and M_2. Draw a tan-

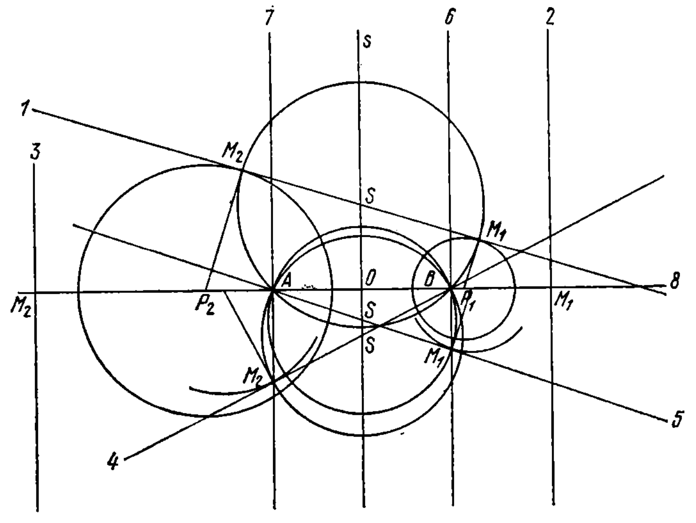

Fig. 6

gent line to the circle at point M_1. Let P_1 be the point of intersection of this tangent and the straight line AB. Construct a circle with centre at the point P_1 and with radius P_1M_1. This is a circle of Apollonius with the limit points A and B. Thus

$$\frac{M_1A}{M_1B} = k_1 > 1.$$

All the points of the straight line l lie outside the circle (C_{k_1}) and so for all points of l different from M_1, the ratio $\frac{MA}{MB}$ is less than k_1, that is, at the point M_1 the ratio $\frac{MA}{MB}$ is a maximum. Similarly it can be proved that this ratio $\frac{MA}{MB}$ is minimal at the point M_2.

Figure 6 gives distinct positions of the straight line l with respect to the points A and B:

(1) The straight line l does not pass through the points A and B and is not perpendicular to AB: at the point M_1 we have the maximum $\left(\text{of the ratio } \frac{MA}{MB}\right)$ and at M_2 the minimum (this case was examined above in detail).

(2) The straight line l is perpendicular to AB, does not pass through the points A and B and is located on the side of the midperpendicular of

AB on which point B lies. At M_1 we have the maximum, and there is no minimum.

(3) The straight line l is perpendicular to AB, does not pass through the points A, B and is located on that side of the midperpendicular of AB on which the point A lies. At point M_2 we have the minimum, and there is no maximum.

(4) The straight line l passes through the point B but does not come to coincidence with AB, the minimum is at M_2, and there is no maximum.

(5) The straight line l passes through point A but does not coincide with AB, the minimum is at point A $\left(\text{the ratio } \frac{MA}{MB} \text{ in this case is zero}\right)$; the maximum is at point M_1.

(6) The straight line l passes through point B and is perpendicular to AB. There is neither maximum nor minimum.

(7) The straight line l passes through point A and is perpendicular to AB. The minimum is at point A $\left(\frac{MA}{MB}=\frac{AA}{AB}=0\right)$, and there is no maximum.

(8) The straight line l coincides with the straight line AB. The minimum is at point A $\left(\frac{MA}{MB}=0\right)$, and there is no maximum.

14°. The locus of points M in the π-plane is found from the condition

$$\frac{MA}{MB}=k.$$

This same locus of points M *in space* is a *sphere S of Apollonius* with limit points A and B. If $0<k<1$, then the sphere S contains within itself the point A, and since A lies in the π-plane, the sphere S and the π-plane intersect along the circle σ, which is the desired locus of points M. But if $k>1$, then the sphere S contains within it the point B and here three cases are possible: either the sphere intersects the π-plane or is tangent to the π-plane, or does not have a single point in common with the π-plane. Let us investigate these cases. On the straight line AB we assume a positive direction from point A to point B. Then

$$\overrightarrow{AP}=\frac{k^2}{k^2-1}\overrightarrow{AB},$$

where P is the centre of the sphere S. The distance d from the centre P of sphere S to the π-plane is

$$d=\frac{k^2}{k^2-1}AB\sin\varphi.$$

The sphere S intersects the π-plane if and only if its radius

$$r=\frac{k}{k^2-1}AB$$

is greater than d:

$$\frac{k\cdot AB}{k^2-1} > \frac{k^2 AB}{k^2-1}\sin\varphi$$

or

$$\sin\varphi < 1/k.$$

When this condition is complied with, the sphere S intersects the π-plane along the circle σ, which is the locus of points M.

If $\sin\varphi = 1/k$, then the sphere S is tangent to the π-plane at the point M, which is the desired locus of points (this point M coincides with the point A if and only if $\varphi = \pi/2$).

Finally, if $\sin\varphi > 1/k$, then the sphere S and the π-plane do not intersect and the desired locus of points M is empty.

Sec. 2. Application of analytic geometry (problems with hints and answers)

1. Plane geometry

1. Fixed in a plane are two distinct points A and B. Given: a real number k. Find the locus of points M for each of which

$$2MA^2 + 3MB^2 = k.$$

Answer. If $k < \frac{5}{6}AB^2$, then the desired locus of points M is empty; if $k = \frac{6}{5}AB^2$, the locus consists of a single point, P, which divides the directed line segment $\overrightarrow{AB}$ in the ratio $3/2$:

$$\frac{\overrightarrow{AP}}{\overrightarrow{PB}} = \frac{3}{2};$$

if $k > \frac{6}{5}AB^2$, the desired locus of points M is a circle with centre at the point P and with radius $R = \frac{1}{5}\sqrt{5k - 6AB^2}$.

2. Fixed in a plane are two distinct points A and B. Given: a real number k. Find the locus of points M for each of which

$$2MA^2 - 3MB^2 = k.$$

Answer. If $k > 6AB^2$, the desired locus of points M is empty; if $k = 6\ AB^2$, the locus of points M consists of the single point P, which divides the directed line segment $\overrightarrow{AB}$ in the ratio $-3/2$:

$$\frac{\overrightarrow{AP}}{\overrightarrow{PB}} = -\frac{3}{2};$$

if $k < 6AB^2$, the desired locus of points M is a circle with centre P and radius $R = \sqrt{6AB^2 - k.}$

3. Fixed in a plane are three points A, B, C. Find the locus of points M for each of which

$$MA^2 + MB^2 = MC^2.$$

Hint. Introduce a rectangular Cartesian system of coordinates.

Answer. If the angle C of the triangle ABC is obtuse, the desired locus of points is empty; if $C = \pi/2$, the locus of points M consists of the single point D symmetric to the point C with respect to the midpoint of AB; if C is an acute angle, the locus is a circle with centre D and radius $\sqrt{a^2 + b^2 - c^2}$ (a, b, c are the lengths of the sides of $\triangle ABC$).

4. The bisector of the interior angle A of $\triangle ABC$ divides the side BC into line segments $BD = 4$, $DC = 2$. Find the lengths of the sides AB and AC if we know that they are given by integers and that the altitude dropped from vertex A to side BC exceeds $\sqrt{10}$.

Solution. It is given that the bisector AD divides the side BC into segments of length 4 and 2; consequently, $BD/DC = 2$, that is $AB/AC = 2$. The locus of points A for which this relation holds is a circle of Apollonius of radius

$$R = \frac{2ak}{|k^2 - 1|},$$

where $2a = CB = 6$, $k = 2$, that is, $R = \frac{6 \cdot 2}{3} = 4$ with centre P lying on the straight line BC and having the coordinate

$$\overrightarrow{OP} = a\frac{k^2 + 1}{k^2 - 1} = 3 \cdot \frac{5}{3} = 5,$$

where O is the midpoint of segment BC. If a rectangular Cartesian system of coordinates is introduced and $\overrightarrow{BC}$ is taken for the x-axis, and the midpoint O of BC as the origin, then the equation of the circle $(P, 4)$ takes the form

$$(x - 5)^2 + y^2 = 16.$$

Let us consider the straight line $y=3$. This line intersects the circle $(P, 4)$ at the points $A_1(5-\sqrt{7}, 3)$, $A_2(5+\sqrt{7}, 3)$. From this it follows that $A_1C = \sqrt{20-4\sqrt{7}} < 3.1$, $A_2C = \sqrt{20+4\sqrt{7}} > 5.5$. Since $AH > \sqrt{10} > 3$, it follows that point A must lie on the lesser arc $\overset{\frown}{A_1A_2}$ of the circle $(P, 4)$; $A_1C < AC < A_2C$, hence $3.1 < AC < 5.5$, and since it is given that the length of AC must be a whole number, it follows that $AC = 4$, then $AB = 8$, or $AC = 5$ and $AB = 10$.

Supplementary questions. Suppose the bisector AD of the interior angle A of $\triangle ABC$ divides the side BC into segments $BD = 4$, and $DC = 2$.

(a) What is the maximum value of the altitude AH? (*Answer.* 4.)

(b) Within what range do the lengths of the sides AB and AC vary? (*Answer.* $2 < AC < 6$, $4 < AB < 12$.)

(c) Prove that AC and AB increase if the point A describes a circle of Apollonius $\frac{AB}{AC} = 2$ from the point D to its diametrically opposite point.

(d) Prove that the length AD of the bisector of angle A varies from 0 to 8.

(e) Given the length m of a median emanating from A. What is a necessary and sufficient condition for the existence of a triangle with given $BD = 4$, $DC = 2$, m? How is such a triangle constructed? (*Answer.* $1 < m < 9$.)

5. Find the images of the circles of Apollonius (see problem 11)

$$\left(x - a\frac{k^2+1}{k^2-1}\right)^2 + y^2 = \frac{4a^2k^2}{(k^2-1)^2}$$

under the inversion $I = (A, AB^2)$:

$$x + a = 4a^2\frac{X+a}{(X+a)^2+Y^2}, \qquad y = 4a^2\frac{Y}{(X+a)^2+Y^2}.$$

Under this inversion, into what do the circles of an elliptic pencil conjugate to the hyperbolic pencil under consideration go?

Answer. (Γ'): $(X-a)^2 + Y^2 = (2a/k)^2$; this is a family of concentric circles (C_k') with common centre $B(a, 0)$; the radius of the circle (C_k') is equal to $2a/k$. The circle E of the elliptic pencil conjugate to the pencil Γ with centre at the point $S(0, s)$ goes into the radical axis of the circle E and the circle of inversion, that is, into the straight line

$$a(X-a) + sY = 0.$$

6. Prove that the equations

$$x^2 + y^2 - 2px + a^2 = 0,$$

$$x^2 + y^2 - 2qy - a^2 = 0,$$

where a is a fixed positive number, and p and q are real parameters, are, respectively, the equations of the hyperbolic pencil of circles and the conjugate elliptic pencil of circles. What are the fundamental points of these pencils? Into what lines do the circles Γ and E go under the inversion $I = (A, 4a^2)$, where $A = (-a, 0)$.

Answer. The fundamental points are $A(-a, 0)$ and $B(a, 0)$. Under the inversion $I = (A, 4a^2)$, the circle Γ goes into the circle

$$(X - a^2) + Y^2 = 4a^2 \frac{p - a}{p + a}$$

with centre at the point $B(a, 0)$ and radius $R' = 2a \sqrt{\frac{p-a}{p+a}}$. Note that only under the values of the parameter p, $p < -a$ and $p \geqslant a$, the equation of Γ is the equation of a real circle.

Under the same inversion, the circle E goes into the straight line $a(X - a) + pY = 0$ passing through point $B(a, 0)$; this is the radical axis of the circle E and of the circle of inversion.

If we consider the circle Γ as a circle of Apollonius, then k and p are connected by the relation $p = a \frac{k^2+1}{k^2-1}$ whence $\frac{p-a}{p+a} = \frac{1}{k^2}$ and, consequently, $R' = 2a \sqrt{\frac{p-a}{p+a}} = \frac{2a}{k}$.

7. Prove that if

$$f \equiv x^2 + y^2 - 2p_1 x + a^2 = 0, \quad \varphi \equiv x^2 + y^2 - 2p_2 x + a^2 = 0$$

are two distinct circles of a hyperbolic pencil ($p_1 \neq p_2$) with limit points $(\pm a, 0)$, then any circle of this pencil may be given by the equation

$$\lambda f + \mu\varphi = 0,$$

where λ and μ are numbers. Conversely, for any λ and μ of which at least one is nonzero, the equation $\lambda f + \mu\varphi = 0$ is the equation of a circle of a hyperbolic pencil (for $\lambda = -\mu \neq 0$, the equation of the midperpendicular of a line segment bounded by limit points).

Consider the special case of $f = x^2 + y^2$, $\varphi = x - a$.

8. State and prove a similar location for two distinct circles

$$f \equiv x^2 + y^2 - 2q_1 y - a^2 = 0, \quad \varphi \equiv x^2 + y^2 - 2q_2 y - a^2 = 0$$

of the elliptic pencil E.

9. Let M be an arbitrary point in the plane. Prove that the difference between the powers of point M with respect to the circles (O) and (O') with centres O and O' is equal to $2(OO') \cdot (KM)$, where K is the orthogonal projection of point M on the radical axis of (O) and (O').

10. AA_1 and BB_1 are medians of $\triangle ABC$; CC_1 is the altitude. The straight lines AA_1, BB_1, CC_1 form $\triangle A_2B_2C_2$. Find the ratio

$$\frac{(A_2B_2C_2)}{(ABC)}.$$

Answer. $\dfrac{(\cot B - \cot A)^2}{3(2\cot B + \cot A)(\cot B + 2\cot A)}$.

11. A', B', C' are the feet of bisectors of the interior angles of $\triangle ABC$. Knowing the lengths a, b, c of the sides BC, CA, AB of $\triangle ABC$, find the ratio

$$\frac{(A'B'C')}{(ABC)}.$$

Answer. $\dfrac{2abc}{(b+c)(c+a)(a+b)}$.

12. A', B', C' are points of tangency of a circle inscribed in $\triangle ABC$ with the sides BC, CA, AB. Prove that

$$\frac{(A'B'C')}{(ABC)} = \frac{r}{2R}.$$

13. A', B', C' are the feet of the altitudes of $\triangle ABC$. Given: the angles A, B, C of the triangle. Find the ratio

$$\frac{(A'B'C')}{ABC}.$$

For what necessary and sufficient condition do we have $\overrightarrow{A'B'C'} \uparrow\downarrow \overrightarrow{ABC}$?

Answer. $2\cos A\cos B\cos C$; $\overrightarrow{A'B'C'} \uparrow\downarrow \overrightarrow{ABC}$ if and only if $\triangle ABC$ is obtuse-angled.

14. Given in the plane three circles (A), (B), (C) each of which lies outside the other two. Let P, Q, R be the centres of internal similarity of the pairs $(B), (C)$; $(C), (A)$; $(A), (B)$. Find

$$\frac{(PQR)}{(ABC)}$$

knowing that the radii of $(A), (B), (C)$ are respectively equal to R_1, R_2, R_3.

Answer. $\dfrac{2R_1R_2R_3}{(R_2+R_3)(R_3+R_1)(R_1+R_2)}$.

15. Given $\triangle ABC$. Through point D lying on the straight line BC are drawn straight lines DP and DQ, which are parallel, respectively, to AB and AC. Prove that $(DQC) + (PBD) = 0$.

16. Through an arbitrary point lying inside $\triangle ABC$ are drawn straight lines parallel to its sides. These lines divide $\triangle ABC$ into six parts, three of which are triangles with areas S_1, S_2, S_3. Find the area of $\triangle ABC$.

Answer. $(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3})^2$.

17. Let A'', B'', C'' be points symmetric to the vertices A, B, C of triangle ABC with respect to the feet of the bisectors of its interior angles. Given: the lengths a, b, c of the sides BC, CA, AB of triangle ABC. Compute the ratio

$$\frac{(A''\,B''\,C'')}{(ABC)}.$$

Answer. $3 + \dfrac{8abc}{(b+c)(c+a)(a+b)}$.

18. The straight line (and also its portion inside the triangle) that is symmetric to the median with respect to the interior-angle bisector emanating from that vertex is termed the *cimedian* of the triangle. Given: the lengths c and b of the sides AB and AC of $\triangle ABC$. In what ratio does the cimedian emanating from vertex A divide the directed line segment $\overrightarrow{BC}$? Prove that the three cimedians of the triangle intersect in one point (the *Lemoine point*).

Answer. $\dfrac{c^2}{b^2}$.

19. In a right triangle, construct the cimedian emanating from the vertex of the right angle.

Answer. The perpendicular dropped from the vertex of the right angle to the hypotenuse.

20. Express, in terms of the lengths a, b, c of the sides BC, CA, AB of $\triangle ABC$, the area of $\triangle PQR$, where P, Q, R are the projections of the centroid G of the triangle on its sides.

Answer. $\dfrac{4(a^2+b^2+c^2)\,S^3}{9a^2\,b^2\,c^2}$, where S is the area of $\triangle ABC$.

21. The bisectors of the interior angles of $\triangle ABC$ intersect the opposite sides BC, CA, AB respectively at the points A', B', C'. The points A'', B'', C'' are symmetric to the points A', B', C' with respect to the respective vertices A, B, C of $\triangle ABC$. Given: the lengths a, b, c of the sides of $\triangle ABC$. Find the ratio

$$\frac{(A''\,B''\,C'')}{(ABC)}.$$

Answer. $6 + \dfrac{2abc}{(b+c)(c+a)(a+b)}$.

22. Let M and N be the midpoints of the medians BD and CE of $\triangle ABC$. The area of $\triangle ABC$ is equal to S. Compute the area of the quadrangle $BMNC$.

Answer. $\dfrac{5}{16}\,S$.

23. The straight lines AO, BO and CO intersect the sides BC, CA, AB of triangle ABC in the points P, Q, R respectively. Prove that

$$\frac{\overrightarrow{OP}}{\overrightarrow{AP}}+\frac{\overrightarrow{OQ}}{\overrightarrow{BQ}}+\frac{\overrightarrow{OR}}{\overrightarrow{CR}}=1.$$

24. Given two straight lines l and m lying on an oriented plane and intersecting in the point A. Through an arbitrary point O not lying either on a line l, or on a line m a straight line n is drawn that intersects l and m in the points B and C respectively. Prove that the sum

$$\frac{1}{(OAB)}+\frac{1}{(OCA)}$$

does not depend on the choice of n.

25. Through the vertex A of a parallelogram $ABCD$ is drawn an arbitrary straight line that intersects the diagonal BD in a point E, and the straight lines BC and CD intersect in points F and G respectively. Prove that the line segment AE is the mean proportional between the lines EF and EG.

26. Let α, β, γ be points symmetric to some single point O with respect to the midpoints of the sides BC, CA, AB of a triangle ABC. Prove that the straight lines $A\alpha$, $B\beta$ and $C\gamma$ pass through the same point P. Also prove that if point O describes some line Γ, then point P describes a line Γ' that is homothetic to line Γ. Where does the homothetic centre lie? What is the homothetic ratio?

Answer. The homothetic centre lies in the centroid M of $\triangle ABC$; $\frac{\overrightarrow{MP}}{\overrightarrow{MO}}=-\frac{1}{2}$.

27. A straight line l intersects the sides BC, CA, AB of a triangle ABC in the points α, β, γ respectively. Let α', β', γ' be points symmetric to the points α, β, γ respectively about the midpoints of the sides BC, CA, AB. Prove that the points α', β', γ' lie on one straight line.

28. Let $A'B'C'$ be a triangle that is obtained if through each vertex of a triangle ABC we draw a straight line parallel to the opposite side; α, β, γ are points taken respectively on the sides BC, CA, AB. Prove that if the straight lines $A\alpha$, $B\beta$, $C\gamma$ pass through a single point, then the straight lines $A'\alpha$, $B'\beta$, $C'\gamma$ also pass through one point.

29. Drawn through the midpoint of each diagonal of a convex quadrangle is a straight line parallel to the other diagonal. The point of intersection of the lines thus drawn is joined to the midpoints of the sides of the given quadrangle. Prove that the quadrangle is thus partitioned into parts of equal size.

30. The points P and Q divide the directed sides $\overrightarrow{BC}$ and $\overrightarrow{CA}$ of $\triangle ABC$ in the given ratios λ and μ. Suppose the straight lines AP and BQ intersect in a point O. Find the ratios:

$$(1)\ \frac{(ABO)}{(ABC)};\quad (2)\ \frac{(OBP)}{(ABC)};\quad (3)\ \frac{(AOQ)}{(ABC)},$$

Answer. (1) $\dfrac{\lambda}{1+\lambda+\lambda\mu}$; (2) $\dfrac{\lambda^2\mu}{(1+\lambda)(1+\lambda+\lambda\mu)}$; (3) $\dfrac{1}{(1+\mu)(1+\lambda+\lambda\mu)}$.

31. A certain point O lies in the plane of a parallelogram $ABCD$ (AC and BD are the diagonals). Prove that:

1°. If the point O lies inside the parallelogram $ABCD$, then the sum of the areas of $\triangle OAB$ and $\triangle OCD$ is equal to the sum of the areas of the triangles OBC and ODA.

2°. No matter where the point O lies in the plane of parallelogram $ABCD$, $\triangle OAC$ is equal in size either to the sum or the difference of $\triangle OAB$ and $\triangle OAD$.

32. Points A', B', C' are taken on the sides BC, CA and AB of $\triangle ABC$. Let A_1, B_1, C_1 and A_2, B_2, C_2 be the images of the points A, B, C respectively under homothetic transformations with the same homothetic ratio and with homothetic centres in the points C', A', B' and B', C', A'. Prove that $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ have one and the same centroid.

33. Let M_1 and M_2 be two arbitrary points lying on side BC of $\triangle ABC$, and let N be an arbitrary point lying on side AB. Denote the points of intersection of the straight lines NM_1 and NM_2 with side AC by P_1 and P_2 and the points of intersection of the straight lines PM_1 and PM_2 (where P is an arbitrary point of the straight line AC) with the side AB by Q_1 and Q_2. Prove that BC, P_1Q_2, and Q_1P_2 belong to the same pencil.

34. Suppose P and Q are two points lying in the plane of $\triangle ABC$; $A_1B_1C_1$ and $A_2B_2C_2$ are triangles symmetric to $\triangle ABC$ about the points P and Q respectively. Let α, β, γ be points of intersection of the straight lines A_1A_2, B_1B_2, C_1C_2 respectively with the lines BC, CA, AB. Prove that the points α, β, γ are collinear (Fig. 7).

Proof. Let α', β', γ' be points of intersection of the straight lines $A\alpha$, $B\beta$, $C\gamma$ with PQ. Then α', β', γ' are the midpoints of the line segments $A\alpha$, $B\beta$, $C\gamma$. These points lie on the sides of $\triangle A'B'C'$, where A', B', C' are the midpoints of the sides BC, CA, AB, respectively. Under the homothetic transformation $(G, -2)$, where G is the point of intersection of the medians of $\triangle ABC$, the points α', β', γ' go into the points $\alpha'', \beta'', \gamma''$, which also lie on one straight line and on the lines BC, CA, AB, respectively. Here, $C\alpha = 2B'\alpha'$, $B\alpha'' = 2B'\alpha'$ and, hence, $C\alpha = B\alpha''$, whence

$$\frac{\overrightarrow{B\alpha}}{\overrightarrow{\alpha C}} = \frac{\overrightarrow{C\alpha''}}{\overrightarrow{\alpha'' B}}$$

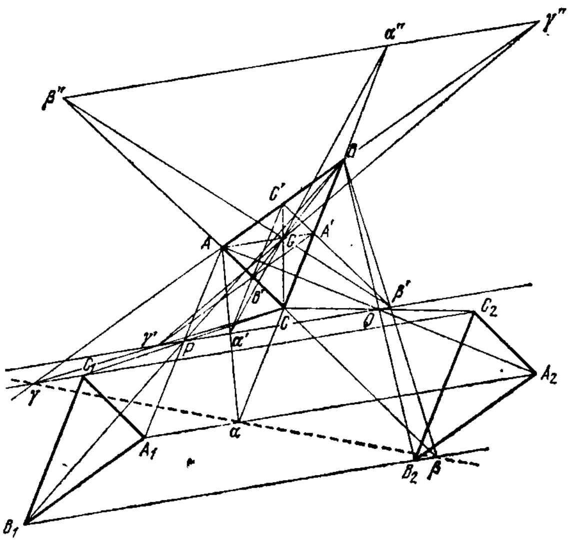

Fig. 7

and similarly for the other two sides. Since the points α'', β'', γ'' are collinear, it follows that

$$\frac{\overrightarrow{C\alpha''}}{\overrightarrow{\alpha''B}} \frac{\overrightarrow{A\beta''}}{\overrightarrow{\beta''C}} \frac{\overrightarrow{B\gamma''}}{\overrightarrow{\gamma''A}} = -1$$

and, hence,

$$\frac{\overrightarrow{B\alpha}}{\overrightarrow{\alpha C}} \frac{\overrightarrow{C\beta}}{\overrightarrow{\beta A}} \frac{\overrightarrow{A\gamma}}{\overrightarrow{\gamma B}} = -1.$$

Consequently, the points α, β, γ are also collinear.

35. Let the straight lines Δ and Δ' intersect the sides of $\triangle ABC$ in the points A_1, B_1, C_1 (line Δ) and A_1', B_1', C_1' (line Δ'), respectively. Prove that the lines A_1B_1', B_1C_1', C_1A_1' intersect, respectively, the straight lines AB, BC, CA in the points P_3, P_1, P_2 lying on one line and the lines $A_1'B_1$, $B_1'C_1$, $C_1'A_1$ intersect the lines AB, BC, CA respectively in the points Q_3, Q_1, Q_2, which also lie on one straight line. The straight lines $P_1P_2P_3$ and $Q_1Q_2Q_3$ are called the *Brocardians* (Brocard lines) of the straight lines Δ and Δ' with respect to $\triangle ABC$.

36. Given: the lengths a, b, c of the sides BC, CA, AB of $\triangle ABC$. Through point C is drawn a bisector CC_0 of the interior angle C. Drawn

through point A is the median AA_0 to the side BC and through point B the altitude BB_0 to side CA. These lines form $\triangle A_1B_1C_1$. Find the ratio

$$\frac{(A_1B_1C_1)}{(ABC)}.$$

Answer. $$\frac{[b(a^2+b^2-c^2)-a(b^2+c^2-a^2)]^2}{(a^2b+b^3-bc^2+2ab^2)(-a^2+3b^2+c^2)(a+2b)}.$$

37. Let $A_1B_1C_1$ and $A_2B_2C_2$ be triangles that are the images of $\triangle ABC$ under the homothetic transformations (P, k), (Q, k). Denote by α, β, γ the points of intersection of the straight lines A_1A_2, B_1B_2, C_1C_2 with the straight lines BC, CA, AB respectively. Find the ratio

$$\frac{(\alpha\beta\gamma)}{(ABC)}.$$

Answer. $-k(k+1)$.

38. Prove that if three diagonals of a hexagon (not necessarily convex) have a common midpoint, then any two of the opposite sides are parallel.

39. From an arbitrary point A_1 lying on side BC of a triangle ABC draw a straight line A_1B_1 parallel to BA up to intersection with CA at point B_1; then draw a straight line B_1C_1 parallel to BC up to intersection with line AB at point C_1 and, finally, draw line C_1A_2 parallel to AC up to intersection with line BC at point A_2. Prove that if A_1 is the midpoint of segment BC, then points A_1 and A_2 coincide; otherwise, continue the process, that is, draw line A_2B_2 parallel to BA, line B_2C_2 parallel to CB, and line C_2A_3 parallel to AC. Prove that the path closes on itself, that is, point A_3 coincides with point A_1.

40. Join vertices A, B, C, D of a parallelogram $ABCD$ with the midpoints of the sides BC, CD, DA, AB so that a parallelogram is formed that lies inside the parallelogram $ABCD$. Prove that the area of the thus formed parallelogram is equal to 1/5 the area of the parallelogram $ABCD$. Another such parallelogram is formed if we join the points A, B, C, D with the midpoints of the sides CD, DA, AB, BC. Prove that the common part of these two small parallelograms is a centrally symmetric octagon and has an area equal to 1/6 that of the parallelogram $ABCD$.

41. $ABCD$ is an arbitrary convex homogeneous lamina. Each of the sides of the quadrangle $ABCD$ is divided into three equal parts: $AA_1 = A_1A_2 = A_2B$, $BB_1 = B_1B_2 = B_2C$, $CC_1 = C_1C_2 = C_2D$, $DD_1 = D_1D_2 = D_2A$. The straight lines A_2B_1, B_2C_1, C_2D_1, D_2A_1 form a parallelogram. Prove that the centroid of the lamina $ABCD$ coincides with the point of intersection of the diagonals of the parallelogram.

Hint. Take the diagonals of the parallelogram for the coordinate axes.

42. Given, with respect to a general Cartesian system of coordinates, two points $M_1(x_1, y_1)$, $M_2(x_2, y_2)$ and the straight line $ax + by + c = 0$. It is given that the points M_1 and M_2 do not lie on the given straight line

and that the third line M_1M_2 intersects the line $ax + by + c = 0$ at some point M. Find the ratio λ in which the point $M(x, y)$ divides the line segment $\overrightarrow{M_1M_2}$.

Answer. $\lambda = -\dfrac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$. *Hint.* $x = \dfrac{x_1 + \lambda x_2}{1 + \lambda}$, $y = \dfrac{y_1 + \lambda y_2}{1 + \lambda}$; these coordinates must satisfy the equation

$$ax + bx + c = 0.$$

43. Taking advantage of the result of the preceding problem, prove that if the straight line l does not pass through any one of the vertices of triangle ABC and intersects its sides BC, CA, AB respectively at the points P, Q, R, then the product of the ratios

$$\lambda = \frac{\overrightarrow{BP}}{\overrightarrow{PC}}, \quad \mu = \frac{\overrightarrow{CQ}}{\overrightarrow{QA}}, \quad \nu = \frac{\overrightarrow{AR}}{\overrightarrow{RB}},$$

at which the points P, Q, R divide the directed line segments $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$, is equal to -1:

$$\lambda\mu\nu = -1.$$

Hint. Introduce into the plane of the triangle ABC a general Cartesian system of coordinates; in this system, let $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C = (x_3, y_3)$ and let $ax + by + c = 0$ be the equation of the straight line l. Then

$$\lambda = -\frac{ax_2 + by_2 + c}{ax_3 + by_3 + c}; \quad \mu = -\frac{ax_3 + bx_3 + c}{ax_1 + by_1 + c}, \quad \nu = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}.$$

Remark. The converse is true as well: if $\lambda\mu\nu = -1$, then the points P, Q, R which divide directed segments $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$ in the ratios λ, μ, ν are collinear. Indeed, suppose the line PQ intersects AB in the point R'. Denote by ν' the ratio in which the point R' divides the directed line segment $\overrightarrow{AB}$. Then $\lambda\mu\nu' = -1$. From this and from the equation $\lambda\mu\nu = -1$ it follows that $\nu = \nu'$, that is,

$$\frac{\overrightarrow{AR}}{\overrightarrow{RB}} = \frac{\overrightarrow{AR'}}{\overrightarrow{R'B}}.$$

Hence, the points R and R' coincide.

44. A second-order curve given by the equation

$$F(x, y) = ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

does not pass through a single vertex of the triangle ABC and intersects the sides BC, CA, AB at the points A_1, A_2; B_1, B_2; C_1, C_2 respectively. Prove that the product of the ratios in which the points A_1, A_2; B_1, B_2; C_1, C_2 respectively divide the directed line segments $\overrightarrow{BC}, \overrightarrow{CA}, \overrightarrow{AB}$:

$$\lambda_1 = \frac{\overrightarrow{BA_1}}{\overrightarrow{A_1C}}, \quad \lambda_2 = \frac{\overrightarrow{BA_2}}{\overrightarrow{A_2C}}, \quad \mu_1 = \frac{\overrightarrow{CB_1}}{\overrightarrow{B_1A}}, \quad \mu_2 = \frac{\overrightarrow{CB_2}}{\overrightarrow{B_2A}}, \quad \nu_1 = \frac{\overrightarrow{AC_1}}{\overrightarrow{C_1B}}, \quad \nu_2 = \frac{\overrightarrow{AC_2}}{\overrightarrow{C_2B}}$$

is equal to 1:

$$\lambda_1 \lambda_2 \mu_1 \mu_2 \nu_1 \nu_2 = 1.$$

Conversely: if on the sides of triangle ABC are chosen points A_1, A_2 (on BC), B_1, B_2 (on CA), C_1, C_2 (on AB) and if the product of the ratios in which these points divide the directed line segments $\overrightarrow{BC}, \overrightarrow{CA}, \overrightarrow{AB}$ is equal to 1, then the points $A_1, A_2, B_1, B_2, C_1, C_2$ lie on one and the same second-order curve.

Hint. The numbers λ_1 and λ_2 are found from the equation

$$F\left(\frac{x_1 + \lambda x_2}{1 + \lambda}, \frac{y_1 + \lambda y_2}{1 + \lambda}\right) = 0,$$

where $B(x_1, y_1)$, $C(x_2, y_2)$ or

$$a\left(\frac{x_1 + \lambda x_2}{1 + \lambda}\right)^2 + 2b\frac{x_1 + \lambda x_2}{1 + \lambda}\frac{y_1 + \lambda y_2}{1 + \lambda} + c\left(\frac{y_1 + \lambda y_2}{1 + \lambda}\right)^2 + 2d\frac{x_1 + \lambda x_2}{1 + \lambda}$$
$$+ 2e\frac{y_1 + \lambda y_2}{1 + \lambda} + f = 0$$

or

$$F(x_2, y_2)\lambda^2 + \ldots + F(x_1, y_1) = 0.$$

By Vieta's theorem,

$$\lambda_1 \lambda_2 = \frac{F(x_1, y_1)}{F(x_2, y_2)}$$

and so on.

The proof of the converse is similar to the proof given in the remark concerning problem 43.

45. Suppose an algebraic curve l of order n does not pass through a single vertex of $\triangle ABC$ and intersects each of the sides of the triangle, BC, CA, AB in n points A_i, B_i, C_i $(i = 1, 2, \ldots, n)$. Then the product of the ratios

$$\frac{\overrightarrow{BA_i}}{\overrightarrow{A_iC}}, \frac{\overrightarrow{CB_i}}{\overrightarrow{B_iA}}, \frac{\overrightarrow{AC_i}}{\overrightarrow{C_iB}}$$

is equal to $(-1)^n$ *(Carnot's theorem)*. Does the converse hold true?

Hint. The proof is similar to that given in the hint referring to problem 44. Generally speaking, the converse for $n > 2$ is not true.

46. Given in the plane are m points P_i $(i = 1, 2, \ldots, m)$. An algebraic curve l of order n does not pass through a single one of the points P_i and intersects the straight lines $P_1P_2, P_2P_3, \ldots, P_{m-1}P_m, P_mP_1$ in n points respectively

$$A_i^{(1)}, A_i^{(2)}, \ldots, A_i^{(n-1)}, A_i^{(n)} \quad (i = 1, 2, \ldots, m).$$

Find the product of the ratios

$$\frac{\overrightarrow{P_1A_i^{(1)}}}{\overrightarrow{A_i^{(1)}P_2}}, \ldots, \frac{\overrightarrow{P_{m-1}A_i^{(n-1)}}}{\overrightarrow{A_i^{(n-1)}P_m}}, \frac{\overrightarrow{P_mA_i^{(n)}}}{\overrightarrow{A_i^{(n)}P_1}}.$$

Answer. $(-1)^{mn}$ (a generalization of Carnot's theorem; see problem 45).

47. A_1, A_2, A_3; B_1, B_2, B_3; C_1, C_2, C_3 are arbitrary points which lie, respectively, on the sides BC, CA, AB of $\triangle ABC$ and are the interior points of the sides. Introduce a general Cartesian coordinate system into the plane of $\triangle ABC$. Since the general equation of a third-order curve contains 10 coefficients, there exists a third-order curve that passes through 9 points A_i, B_i, C_i $(i = 1, 2, 3)$. By the Carnot theorem, the product of the ratios in which the points A_i, B_i, C_i divide $\overrightarrow{BC}, \overrightarrow{CA}, \overrightarrow{AB}$, respectively, is equal to $(-1)^3 = -1$, and yet all these ratios are positive and their product cannot be a negative number. Wherein lies the error?

2. *Solid geometry*

1. $A_1A_2A_3A_4$ is an arbitrary tetrahedron; B_1, B_2, B_3, B_4 are the centroids of its faces $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_4$, $A_1A_2A_3$; $C_{12}, C_{13}, C_{14}, C_{23}, C_{24}, C_{34}$ are midpoints of its edges $A_1A_2, A_1A_3, A_1A_4, A_2A_3, A_2A_4, A_3A_4$. The straight lines A_1B_1, A_2B_2, A_3B_3, A_4B_4 (and also the line segments themselves) are called *medians of the tetrahedron* $A_1A_2A_3A_4$. The straight lines $C_{12}C_{34}$, $C_{13}C_{24}$, $C_{23}C_{14}$ (and also the line segments themselves) are termed *bimedians of the tetrahedron* $A_1A_2A_3A_4$.

Prove that four medians and three bimedians of the tetrahedron $A_1A_2A_3A_4$ pass through one and the same point G, called the *centroid of the tetrahedron* $A_1A_2A_3A_4$; here, $\overrightarrow{A_iG} : \overrightarrow{GB_i} = 3 : 1$ and the bimedians are bisected by the point G. Assuming the radius vectors of the points A_i are equal to $\overrightarrow{OA_i} = \mathbf{r}_i$ $(i = 1, 2, 3, 4)$, find the radius vectors $\mathbf{r}_{ij}$ of the midpoints of the edges A_iA_j, the radius vectors $\mathbf{r}_{ijk}$ of the centroids of the faces $A_iA_jA_k$, and the radius vector $\mathbf{r}$ of the centroid G.

Answer. $\mathbf{r}_{ij} = \dfrac{\mathbf{r}_i + \mathbf{r}_j}{2}$, $\mathbf{r}_{ijk} = \dfrac{\mathbf{r}_i + \mathbf{r}_j + \mathbf{r}_k}{3}$, $\mathbf{r} = \dfrac{\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4}{4}$.

2. Prove that the six planes passing through the edges A_iA_j of the tetrahedron $A_1A_2A_3A_4$ and the midpoints of the opposite edges pass through the centroid G of the tetrahedron.

Hint. $A_iA_jC_{kl}$ and $A_kA_lC_{ij}$ intersect along the bimedian $C_{kl}C_{ij}$.

3. Prove that the six planes that pass through the edges of the tetrahedron and divide its volume into two intersect in a single point.

4. OA, OB, OC are the edges of a parallelepiped; A', B', C', O' are the vertices symmetric to the vertices A, B, C, O about the centre of the parallelepiped. Prove the following statements:

1°. The diagonal OO' of the parallelepiped is divided into three equal parts by the planes ABC and $A'B'C'$.

2°. The diagonal OO' intersects the planes of the triangles ABC and $A'B'C'$ in their centroids.

Hint. Set $O = (0, 0, 0)$, $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$.

5. Let G_a, G_b, G_c, G_d be the centroids of the faces BCD, CDA, ABD, ABC of the tetrahedron $ABCD$. Prove the following statements:

1°. If a straight line λ intersects the faces BCD, CDA, ADB, ABC in the points $\omega_a, \omega_b, \omega_c, \omega_d$, respectively, then the midpoints of the segments $A\omega_a, B\omega_b, C\omega_c, D\omega_d$ are coplanar.

2°. The tetrahedrons $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are images of the tetrahedron $ABCD$ under the homothetic transformations $(P_1, -1/2), (P_2, -1/2)$, where P_1 and P_2 are arbitrary points. Prove that the straight lines A_1A_2, B_1B_2, C_1C_2, D_1D_2 intersect the faces BCD, CDA, DAB, ABC in the points $\alpha, \beta, \gamma, \delta$ lying in one plane.

Proof. Let G be the centroid of the tetrahedron $ABCD$; let $\alpha', \beta', \gamma', \delta'$ be the points of intersection of the straight line P_1P_2 with the straight lines $A\alpha, B\beta, C\gamma, D\delta$; let $\omega_a, \omega_b, \omega_c, \omega_d$ be points at which the following lines intersect: $G\alpha$ and $G_a\alpha'$; $G\beta$ and $G_b\beta'$; $G\gamma$ and $G_c\gamma'$; $G\delta$ and $G_d\delta'$. The points $\alpha', \beta', \gamma', \delta'$ divide the directed line segments $\overrightarrow{A\alpha}, \overrightarrow{B\beta}, \overrightarrow{C\gamma}, \overrightarrow{D\delta}$ in the ratio of 2 : 1 and therefore lie in the planes of the faces of the tetrahedron $G_aG_bG_cG_d$. The point G divides segment $\overrightarrow{AG_a}$ in the ratio of 3 : 1, and therefore ω_a is the midpoint of segment $G_a\alpha'$ and $\overrightarrow{G\alpha} = 3\overrightarrow{G\omega_a}$. On the basis of item 1°, the points $\omega_a, \omega_b, \omega_c, \omega_d$ lie in one plane, and hence so also do the points $\alpha, \beta, \gamma, \delta$ obtained from the points $\omega_a, \omega_b, \omega_c, \omega_d$ under the homothetic transformation $(G, 3)$.

3°. The images $\omega_a', \omega_b', \omega_c', \omega_d'$ of the points $\omega_a, \omega_b, \omega_c, \omega_d$ under the homothetic transformations $(G_a, -1/2), (G_b, -1/2), (G_c, -1/2), (G_d, -1/2)$ are coplanar.

Hint. See item 2°, $\dfrac{\overrightarrow{\alpha A'}}{\overrightarrow{A'G_a}} = -\dfrac{\overrightarrow{AG}}{\overrightarrow{GG_a}}\,\dfrac{\overrightarrow{\alpha\alpha'}}{\overrightarrow{\alpha' A}} = -\dfrac{3}{2}\ \left(\dfrac{\overrightarrow{G_a\alpha}}{\overrightarrow{G_aA'}} = -\dfrac{1}{2}\right)$.

6. Given: four arbitrary points A', B', C', D' lying, respectively, in the faces BCD, CDA, DAB, ABC of a tetrahedron $ABCD$. Let (A_1, B_1, C_1, D_1),

(A_2, B_2, C_2, D_2), (A_3, B_3, C_3, D_3) be images of the points A, B, C, D under a homothetic transformation with ratio k and, respectively, with centres (D', A', B', C'), (C', D', A', B'), (B', C', D', A') so that

$$\frac{\overrightarrow{D'A_1}}{\overrightarrow{D'A}} = k, \frac{\overrightarrow{A'B_1}}{\overrightarrow{A'B}} = k, \frac{\overrightarrow{B'C_1}}{\overrightarrow{B'C}} = k, \frac{\overrightarrow{C'D_1}}{\overrightarrow{C'D}} = k, \frac{\overrightarrow{C'A_2}}{\overrightarrow{C'A}} = k \text{ and so forth.}$$

Prove that the tetrahedrons $A_1B_1C_1D_1$, $A_2B_2C_2D_2$, $A_3B_3C_3D_3$ have a common centroid.

7. Let P be an arbitrary point lying inside a tetrahedron $ABCD$. Denote by A', B', C', D' the points of intersection of the straight lines PA, PB, PC, PD with the opposite faces. Prove that

$$\frac{AP}{PA'} + \frac{BP}{PB'} + \frac{CP}{PC'} + \frac{DP}{PD'} \geqslant 12,$$

$$\frac{AP}{PA'}\frac{BP}{PB'}\frac{CP}{PC'}\frac{DP}{PD'} \geqslant 81.$$

Under what condition do we have equality?

8. Let a and a', b and b', c and c' be the respective points in which the edges BC and AD, CA and DB, AB and DC of a tetrahedron $ABCD$ intersect an arbitrary plane. Denote by G_a, G_b, C_c, G_d the centroids of the triangles $a'bc$, $b'ca$, $c'ab$, $a'b'c'$. Construct the points A', B', C', D' such that

$$\overrightarrow{AA'} = 3\overrightarrow{AG_a},\ \overrightarrow{BB'} = 3\overrightarrow{BG_b},\ \overrightarrow{CC'} = 3\overrightarrow{CG_c},\ \overrightarrow{DD'} = 3\overrightarrow{DG_d}.$$

Prove that the points A', B', C', D' lie in a single plane.

9. $T = ABCD$ is an arbitrary tetrahedron; $\alpha\beta\gamma\delta$ is spatial quadrangle whose sides are equal and parallel to the medians of the given tetrahedron. Through the midpoints of the sides of the quadrangle $\alpha\beta\gamma\delta$ draw planes parallel respectively to the faces of the tetrahedron (so that if $\alpha\beta \mathrel{\#} AA'$, where AA' is a median of the tetrahedron $ABCD$, then the plane passing through the midpoint of segment $\alpha\beta$ is parallel to the plane BCD and so on). The four planes thus drawn form a tetrahedron T_1. Prove that the volume of T_1 is eight times that of T and that the tetrahedrons T_1 and $\alpha\beta\gamma\delta$ have a common centroid.

10. Draw through the vertices A', B', C', D' of a tetrahedron $A'B'C'D'$ parallel lines d_1, d_2, d_3, d_4. Let $ABCD$ be a tetrahedron homothetic to the tetrahedron $A'B'\mathrm{B}'D'$ under a homothetic transformation with ratio k. Denote by A_1, B_1, C_1, D_1 the points of intersection of the lines d_1, d_2, d_3, d_4 respectively with the planes BCD, CDA, ABD, ABC. Prove that

$$\overline{A_1B_1C_1D_1} = -k^2(2k+1)\,\overline{ABCD}.$$

11. Through the vertices A, B, C, D of a tetrahedron $ABCD$ draw planes a, b, c, d parallel to some plane m. Through the vertices A', B', C', D'

of the tetrahedron $A'B'C'D'$ draw planes a', b', c', d' parallel to some plane m'. Here the planes m and m' are chosen so that the straight lines (a, a'), (b, b'), (c, c'), (d, d') lie in one plane (the P-plane). Prove that when the planes m and m' are changed (however, the change must be such that coplanarity of the indicated four lines is preserved at all times), the P-plane will rotate about a fixed point.

12. Straight lines joining the vertices A, B, C, D of a tetrahedron $ABCD$ with point P intersect its opposite faces in the points A', B', C', D'. On segments $\overrightarrow{AA'}$, $\overrightarrow{BB'}$, $\overrightarrow{CC'}$, $\overrightarrow{DD'}$ consider points A_1, B_1, C_1, D_1 such that

$$\frac{\overrightarrow{AA'}}{\overrightarrow{A'A_1}}=\frac{\overrightarrow{BB'}}{\overrightarrow{B'B_1}}=\frac{\overrightarrow{CC'}}{\overrightarrow{C'C_1}}=\frac{\overrightarrow{DD'}}{\overrightarrow{D'D_1}}=k$$

and the points A_2, A_3, A_4 in which the straight lines A_1B_1, A_1C_1, A_1D_1 intersect the plane BCD. Prove that the areas of the triangles A_2CD, A_3DB, A_4BC are equal. Consider the case $k = 2$.

13. Given a tetrahedron $T = ABCD$ and four straight lines a, b, c, d that pass through one and the same point P and are parallel to the four given lines. Let line a intersect the planes BCD, CDA, DAB, ABC in the points l_1, m_4, n_3, p_2; line b in the points l_2, m_1, n_4, p_3, line c in the points l_3, m_2, n_1, p_4, and line d in the points l_4, m_3, n_2, p_1. Prove that, generally, there exists only one position of point P under which the quadruplets of points (l_1, m_1, n_1, p_1) (l_2, m_2, n_2, p_2), (l_3, m_3, n_3, p_3), (l_4, m_4, n_4, p_4) are coplanar. Consider the case where the straight lines a, b, c, d are parallel to the medians of the tetrahedron $ABCD$.

CHAPTER III

THE USE OF COMPLEX NUMBERS IN PLANE GEOMETRY

Sec. 1. Solved problems

Before examining the problems, make a brief study of the theoretical material given in Chapter V, Sec. 4.

Since the method applied here is not readily available and is effective and simple in the solution of many problems of plane geometry, we have selected a large number of problems and have provided solutions; there are also exercises in the unaided solving of problems (with hints and answers)

Problem 1. ABC is an arbitrary triangle, G is the point of intersection of its medians (centroid); H is the point of intersection of the altitudes (orthocentre); O is the centre of a circumscribed circle $(O) = (ABC)$; A_1, B_1, C_1 are the midpoints of the sides BC, CA, AB respectively; A_2, B_2, C_2 are the feet of the altitudes; A_3, B_3, C_3 are the corresponding midpoints of the line segments AH, BH, CH (Euler's points); A_4, B_4, C_4 are the corresponding points symmetric to the orthocentre H about the straight lines BC, CA, AB.

Prove that:

1°. The points O, G, H are collinear and $\overrightarrow{OH} = 3\overrightarrow{OG}$.

2°. The points $A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3$ lie on one and the same circle (O_9) *(Euler's circle* or *the nine-point circle of* $\triangle$ *ABC)*. The centre of the circle O_9 is the midpoint of the line OH, and the radius of the circle (O_9) is equal to $R/2$, where R is the radius of the circle circumscribed about the triangle ABC.

3°. The points A_4, B_4, C_4 lie on the circumscribed circle.

Solution 1°. Take (ABC) for the unit circle. Let a, b, c be the corresponding affixes of the points A, B, C. Prove that $a + b + c$ is the affix of the orthocentre H of triangle ABC.

The slope of the straight line BC is

$$\varkappa = \frac{c-b}{\bar{c}-\bar{b}} = \frac{c-b}{\frac{1}{c}-\frac{1}{b}} = -bc.$$

The slope of the line joining vertex $A(a)$ and point M and having the affix $a + b + c$ is

$$\varkappa' = \frac{b+c}{\bar{b}+\bar{c}} = \frac{b+c}{\frac{1}{b}+\frac{1}{c}} = bc,$$

whence

$$\varkappa + \varkappa' = 0.$$

Hence, $AM \perp BC$. In similar fashion it is proved that $BM \perp CA$ and $CM \perp AB$. Hence, M—the point with affix $a + b + c$—coincides with point H (the point of intersection of the altitudes of $\triangle ABC$).

Furthermore, the affix of point G is

$$\frac{a + b + c}{3},$$

whence and also from the fact that the affix of point H is equal to $a + b + c$ it follows that

$$\overrightarrow{OH} = 3\overrightarrow{OG}.$$

That is, the points O, G, H are collinear.

2°. The midpoints A_1, B_1, C_1 of the sides BC, CA, AB have affixes

$$a_1 = \frac{b + c}{2}, \quad b_1 = \frac{c + a}{2}, \quad c_1 = \frac{a + b}{2}.$$

Let E be the midpoint of line OH. Its affix is

$$\varepsilon = \frac{a + b + c}{2}.$$

Since

$$\varepsilon - a_1 = a/2, \quad \varepsilon - b_1 = b/2, \quad \varepsilon - c_1 = c/2$$

and $|a| = |b| = |c| = 1$, it follows that

$$|\varepsilon - a_1| = |\varepsilon - b_1| = |\varepsilon - c_1| = 1/2 = R/2 \quad (R = 1)$$

that is,

$$EA_1 = EB_1 = EC_1 = R/2.$$

Thus, the midpoints A_1, B_1, C_1 of lines BC, CA, AB are at equal distances—$R/2$—from the midpoint of line OH and, hence, lie on circle $(A_1B_1C_1)$ with centre $E((a + b + c)/2)$ and radius $R/2$.

Furthermore, the equations of the straight lines AH and BC are of the form

$$z - a = bc(\bar{z} - \bar{a}),$$

$$z - b = -bc(\bar{z} - \bar{b})$$

or

$$z - bc\bar{z} = a - \bar{a}bc,$$

$$z + bc\bar{z} = b + c.$$

Forming the half-sum of these equations, we find the affix $z=a_2$ of point A_2:

$$a_2 = \frac{a+b+c}{2} - \frac{bc}{2a} = \varepsilon - \frac{bc}{2a},$$

whence

$$EA_2 = |a_2 - \varepsilon| = \left| -\frac{bc}{2a} \right| = \frac{1}{2} = \frac{R}{2}$$

($|a| = |b| = |c| = 1$). Similarly, proof is given that

$$EB_2 = R/2, \quad EC_2 = R/2.$$

Thus, the points $A_1, B_1, C_1, A_2, B_2, C_2$ lie on a circle whose centre is the midpoint of line OH and whose radius is equal to $R/2$.

Furthermore, the affixes of the midpoints A_3, B_3, C_3 of lines AH, BH, CH are, respectively,

$$a_3 = \frac{a+a+b+c}{2} = \varepsilon + \frac{a}{2},$$

$$b_3 = \varepsilon + \frac{b}{2},$$

$$c_3 = \varepsilon + \frac{c}{2},$$

whence

$$EA_3 = |a_3 - \varepsilon| = |a/2| = 1/2 = R/2$$

and, similarly,

$$EB_3 = EC_3 = R/2.$$

Thus, all nine points A_k, B_k, C_k $(k = 1, 2, 3)$ lie on the circle $(E, R/2)$' whose centre E is the midpoint of line OH and whose radius is $R/2$.

3°. Let us now go back to the equation of the straight line AH,

$$z - a = bc(\bar{z} - \bar{a}),$$

that passes through point A perpendicular to BC. Now find the affixes of the points of intersection of this line with the unit circle $z\bar{z} = 1$. Provided $|z| = 1$, we have

$$z - a = bc\left(\frac{1}{z} - \frac{1}{a}\right)$$

or

$$z - a = -bc\,\frac{z-a}{az}.$$

One of the roots of this equation is $z = a$ (the affix of point A); the other is

$$z = -\frac{bc}{a}$$

$\Big($the fact that point N with this affix belongs to the unit circle follows from the equality $\left| -\frac{bc}{a} \right| = 1\Big)$.

We will prove that the midpoint of line HN coincides with point A_2. Indeed, the affix of this midpoint is

$$\frac{a + b + c - \frac{bc}{a}}{2} = \varepsilon - \frac{bc}{2a} = a_2.$$

Thus, point N coincides with point A_4.

In similar fashion, proof is given that the altitudes emanating from vertices B and C intersect the circle (ABC) in the points B_4 and C_4, which are symmetric to the orthocentre H with respect to the sides CA and AB.

Problem 2. (Boutain points). Suppose $\triangle ABC$ is inscribed in the unit circle (O). Let z_1, z_2, z_3 be the affixes of the points A, B, C. A *Boutain point* for $\triangle ABC$ is a point such that when it is chosen as the unit point (that is, a point whose affix is $z = 1$), the equation $\sigma_3 = z_1z_2z_3 = 1$ holds true. Prove that for a given circle (O) and $\triangle ABC$ inscribed in it, there are three Boutain points on the circle (O) that form an equilateral triangle.

Proof. Take the point α ($|\alpha| = 1$) for the new unit point. Then the new affixes of the points A, B, C will be $\frac{z_1}{\alpha}$, $\frac{z_2}{\alpha}$, $\frac{z_3}{\alpha}$. The point α will be the Boutain point if the product of these affixes is equal to 1:

$$\frac{z_1z_2z_3}{\alpha^3} = 1 \text{ or } \alpha^3 = \sigma_3.$$

This equation has three roots:

$$\sqrt[3]{\sigma_3}, \quad \varepsilon\sqrt[3]{\sigma_3}, \quad \varepsilon^2\sqrt[3]{\sigma_3},$$

where $\sqrt[3]{\sigma_3}$ is any one of three values of the cubic root of σ_3, and ε is any one of two imaginary values $\sqrt[3]{1}$, that is,

$$\text{either} \quad \varepsilon = \frac{-1 + i\sqrt{3}}{2} \quad \text{or} \quad \varepsilon = \frac{-1 - i\sqrt{3}}{2}.$$

For example, let

$$\varepsilon = \frac{-1 + i\sqrt{3}}{2} = \cos 120^\circ + i \sin 120^\circ,$$

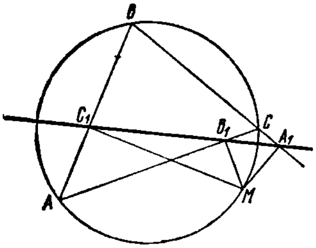

Fig. 8

then

$$\varepsilon^2 = \cos 240° + i \sin 240°,$$

and to construct the Boutain points it is necessary first to construct the point for the chosen value $\sqrt[3]{\sigma_3}$, then multiply it by ε (a rotation through 120°) and then once again by ε (another rotation through 120°). We obtain the vertices of an equilateral triangle with affixes

$$\sqrt[3]{\sigma_3},\quad \varepsilon\sqrt[3]{\sigma_3},\quad \varepsilon^2\sqrt[3]{\sigma_3}.$$

Problem 3. Given an arbitrary triangle ABC. Take on the circle (ABC) an arbitrary point M (Fig. 8). Let A_1, B_1 and C_1 be the orthogonal projections of M on the straight lines BC, CA, and AB. Prove that the points A_1, B_1, C_1 are collinear. Taking the circle (ABC) for the unit circle and assuming that the affixes of the points A, B, C, M are equal, respectively, to z_1, z_2, z_3, z_0, set up the equation of the straight line $A_1B_1C_1$ (the *Simson line* for the point M with respect to the triangle ABC).

Solution. The equation of the line BC and the straight line passing through point M perpendicularly to BC is of the form

$$z - z_2 = -z_2z_3(\bar{z} - \bar{z}_2),$$
$$z - z_0 = z_2z_3(\bar{z} - \bar{z}_0)$$

or

$$z + z_2z_3\bar{z} = z_2 + z_3,$$
$$z - z_2z_3\bar{z} = z_0 - z_2z_3\bar{z}_0.$$

Forming the half-sum of these equations, we find the affix $z = a_1$ of the orthogonal projection of point M on the straight line BC:

$$a_1 = \frac{1}{2}(z_0 + z_2 + z_3 - z_2z_3\bar{z}_0). \qquad (*)$$

Similarly we can find the affix b_1 of point B_1:

$$b_1 = \frac{1}{2}(z_0 + z_3 + z_1 - z_3z_1\bar{z}_0).$$

Let us find the slope of the straight line A_1B_1; we have

$$a_1 - b_1 = \frac{1}{2}[z_2 - z_1 - z_3\bar{z}_0(z_2 - z_1)]$$

$$= \frac{1}{2}(z_2 - z_1)(1 - z_3\bar{z}_0) = \frac{1}{2}(z_2 - z_1)\left(1 - \frac{z_3}{z_0}\right) = \frac{1}{2}\,\frac{(z_2 - z_1)(z_0 - z_3)}{z_0};$$

$$\bar{a}_1 - \bar{b}_1 = \frac{z_0}{2}\left(\frac{1}{z_2} - \frac{1}{z_1}\right)\left(\frac{1}{z_0} - \frac{1}{z_3}\right) = \frac{1}{2}(z_2 - z_1)(z_0 - z_3)\frac{1}{z_1z_2z_3}$$

and so

$$\varkappa = \frac{a_1 - b_1}{\bar{a}_1 - \bar{b}_1} = \frac{\sigma_3}{z_0} = \sigma_3 \bar{z}_0.$$

The equation of the straight line A_1B_1 may be written as

$$z - a_1 = \sigma_3 \bar{z}_0(\bar{z} - \bar{a}_1)$$

or, taking into account equation (*),

$$z - \frac{1}{2}(z_0 + z_2 + z_3 - z_2 z_3 \bar{z}_0)$$

$$= \sigma_3 \bar{z}_0 = \left[\bar{z} - \frac{1}{2}\left(\frac{1}{z_0} + \frac{1}{z_2} + \frac{1}{z_3} - \frac{z_0}{z_2 z_3}\right)\right].$$

This equation can be simplified:

$$z - \sigma_3 \bar{z}_0 \bar{z} = \frac{1}{2}\left((z_0 + z_2 + z_3 - \frac{z_2 z_3}{z_0}\right)$$

$$- \frac{z_1 z_2 z_3}{2 z_0}\left(\frac{1}{z_0} + \frac{1}{z_2} + \frac{1}{z_3} - \frac{z_0}{z_2 z_3}\right)$$

or

$$z - \sigma_3 \bar{z}_0 \bar{z} = \frac{1}{2}\left(z_0 + z_2 + z_3 - \frac{z_2 z_3}{z_0} - \frac{\sigma_3}{z_0^2} - \frac{z_1 z_3}{z_0} - \frac{z_1 z_2}{z_0} + z_1\right)$$

or

$$z - \sigma_3 \bar{z}_0 \bar{z} = \frac{1}{2}(z_0 + \sigma_1 - \sigma_2 \bar{z}_0 - \sigma_3 \bar{z}_0^2), \tag{1}$$

where

$$\sigma_1 = z_1 + z_2 + z_3, \quad \sigma_2 = z_1 z_2 + z_2 z_3 + z_3 z_1, \quad \sigma_3 = z_1 z_2 z_3.$$

The symmetry of this equation with respect to z_1, z_2, z_3 permits asserting that line A_1B_1 also passes through point C_1. Incidentally, this becomes evident by substituting into the left and right members of equation (1) the affix

$$c_1 = \frac{1}{2}(z_0 + z_1 + z_2 - z_1 z_2 \bar{z}_0)$$

of point C_1 (the results will be the same; check this!) or by seeing that the following equation holds:

$$\begin{vmatrix} a_1 & \bar{a}_1 & 1 \\ b_1 & \bar{b}_1 & 1 \\ c_1 & \bar{c}_1 & 1 \end{vmatrix} = 0$$

(this is left to the reader to verify).

If z_0 is the unit point, then the equation of the Simson line (1) assumes the form

$$z - \sigma_3\bar{z} = \frac{1}{2}(1 + \sigma_1 - \sigma_2 - \sigma_3), \tag{2}$$

and if we take the Boutain point for the unit point (that is, $\sigma_3 = 1$), then

$$z - \bar{z} = \frac{1}{2}(\sigma_1 - \sigma_2). \tag{3}$$

Remark. It will be proved below that if point M does not lie on the circle (ABC), then its projections A_1, B_1, C_1 on the sides BC, CA, AB do not lie on one straight line.

Problem 4. Prove that if for the unit point we take the Boutain point M with respect to $\triangle ABC$ inscribed in the unit circle (O), then the Simson line corresponding to point M will be collinear with the diameter of the unit circle passing through M (see problems 2 and 3).

Solution. The equation of the Simson line will have the form

$$z - \bar{z} = \frac{1}{2}(\sigma_1 - \sigma_2), \tag{3}$$

which means it will be collinear with the x-axis or the diameter of the unit circle passing through point M because M is the unit point of the x-axis.

Note the converse: if the Simson line for point M of (ABC) with respect to $\triangle ABC$ is collinear with the diameter of the circle (the diameter passing through M), then M is the Boutain point.

True enough, for if M is the unit point, then the slope of the Simson line is equal to σ_3 and if the Simson line is parallel to the diameter passing through M, that is, parallel to the x-axis, then $\sigma_3 = 1$, since the slope of the x-axis is equal to 1.

Problem 5. Let us consider $\triangle ABC$ inscribed in the unit circle (O). Prove that:

1°. The point P with affix σ_2 is symmetric to the orthocentre H of $\triangle ABC$ with respect to the diameter δ of the unit circle, which diameter is parallel to the Simson line constructed for the unit point with respect to $\triangle ABC$.

2°. The point Q with affix σ_3 is symmetric to the unit point with respect to the diameter δ.

Solution. 1°. The equation of the diameter δ is of the form

$$z - \sigma_3\bar{z} = 0$$

[see equation (2) of problem 3]. The affix of the projection of the orthocentre H on this diameter is found from the system of equations

$$z - \sigma_3\bar{z} = 0,$$

$$z - \sigma_1 = -\sigma_3(\bar{z} - \bar{\sigma}_1)$$

or

$$z - \sigma_3\bar{z} = 0,$$
$$z + \sigma_3\bar{z} = \sigma_1 + \sigma_3\bar{\sigma}_1$$

or, since $\bar{\sigma}_1 = \frac{\sigma_2}{\sigma_3}$, it follows that

$$z - \sigma_3\bar{z} = 0,$$
$$z + \sigma_3\bar{z} = \sigma_1 + \sigma_2.$$

Adding these equations, we find the affix λ of the projection of the orthocentre H on the diameter δ:

$$\lambda = \frac{\sigma_1 + \sigma_2}{2}.$$

The affix ω of the point symmetric to point H with respect to the diameter δ is found from the equation

$$\frac{\omega + \sigma_1}{2} = \frac{\sigma_1 + \sigma_2}{2},$$

whence

$$\omega = \sigma_2.$$

2°. The equation of the perpendicular dropped from the unit point to the diameter δ is of the form

$$z - 1 = -\sigma_3(\bar{z} - 1).$$

Solving this equation together with the equation of the unit circle $z\bar{z} = 1$, we obtain

$$z - 1 = -\sigma_3\left(\frac{1}{z} - 1\right),$$
$$z - 1 = \sigma_3\frac{z-1}{z}.$$

The roots of this equation are: $z = 1$, $z = \sigma_3$.

Problem 6. 1°. Let A_1, B_1, C_1 be the orthogonal projections of point P on the sides BC, CA, AB of $\triangle ABC$. We take the centre O of (ABC) for the origin. Let Rz_1, Rz_2, Rz_3 be the affixes of the points A, B, C, where R is the radius of the circle $(ABC) = (|z_1| = |z_2| = |z_3| = 1)$. Let p be the affix of point P. Find the affixes a_1, b_1, c_1 of points A_1, B_1, C_1 and then express the area $(A_1B_1C_1)$ of the oriented $\overrightarrow{\triangle A_1B_1C_1}$ in terms of the area (ABC) of the oriented $\overrightarrow{\triangle ABC}$, in terms of the lengths a, b, c of the sides BC, CA, AB and in terms of the affix p of point P.

Consider the following special cases:

2°. The point P coincides with the centroid G of $\triangle ABC$.

3°. The point P coincides with the centre I of the circle inscribed in $\triangle ABC$.

4°. The point P coincides with the centre O of circle (ABC).

5°. The point P coincides with the orthocentre H of $\triangle ABC$.

Solution 1°. The equation of line BC is

$$z - z_2R = -z_2z_3(\bar{z} - \bar{z}_2R) \tag{4}$$

or

$$z + z_2z_3\bar{z} = R(z_2 + z_3)$$

or

$$z + z_2z_3\bar{z} = R(\sigma_1 - z_1). \tag{5}$$

The equation of the perpendicular dropped from point P to line BC is of the form

$$z - p = z_2z_3(\bar{z} - \bar{p})$$

or

$$z - z_2z_3\bar{z} = p - z_2z_3\bar{p}$$

or

$$z - z_2z_3\bar{z} = p - \bar{z}_1\sigma_3\bar{p}. \tag{6}$$

Forming the half-sum of equations (5) and (6), we can find the affix $z = a_1$ of point A_1:

$$a_1 = \frac{1}{2}(R\sigma_1 + p - Rz_1 - \bar{z}_1\sigma_3\bar{p}).$$

Similarly,

$$b_1 = \frac{1}{2}(R\sigma_1 + p - Rz_2 - \bar{z}_2\sigma_3\bar{p}),$$

$$c_1 = \frac{1}{2}(R\sigma_1 + p - Rz_3 - \bar{z}_3\sigma_3\bar{p}).$$

From this we get

$$(A_1B_1C_1) = \frac{i}{4}\begin{vmatrix} \frac{1}{2}(R\sigma_1 + p - Rz_1 - \bar{z}_1\sigma_3\bar{p}) & \frac{1}{2}(R\bar{\sigma}_1 + \bar{p} - R\bar{z}_1 - z_1\bar{\sigma}_3p) & 1 \\ \frac{1}{2}(R\sigma_1 + p - Rz_2 - \bar{z}_2\sigma_3\bar{p}) & \frac{1}{2}(R\bar{\sigma}_1 + \bar{p} - R\bar{z}_2 - z_2\bar{\sigma}_3p) & 1 \\ \frac{1}{2}(R\sigma_1 + p - Rz_3 - \bar{z}_3\sigma_3\bar{p}) & \frac{1}{2}(R\bar{\sigma}_1 + \bar{p} - R\bar{z}_3 - z_3\bar{\sigma}_3p) & 1 \end{vmatrix} =$$

$$= \frac{i}{16}\begin{vmatrix} Rz_1 + \bar{z}_1\sigma_3\bar{p} & R\bar{z}_1 + z_1\bar{\sigma}_3 p & 1 \\ Rz_2 + \bar{z}_2\sigma_3\bar{p} & R\bar{z}_2 + z_2\bar{\sigma}_3 p & 1 \\ Rz_3 + \bar{z}_3\sigma_3\bar{p} & R\bar{z}_3 + z_3\bar{\sigma}_3 p & 1 \end{vmatrix}$$

$$= \frac{i}{16}\left(R^2\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} + p\bar{p}\begin{vmatrix} \bar{z}_1 & z_1 & 1 \\ \bar{z}_2 & z_2 & 1 \\ \bar{z}_3 & z_3 & 1 \end{vmatrix}\right)$$

$$= \frac{i}{16}\left(-4i(ABC) + \frac{4ip\bar{p}}{R^2}(ABC)\right)$$

$$= \frac{1}{4}(ABC)\left(1 - \frac{p\bar{p}}{R^2}\right) = \frac{R^2 - p\bar{p}}{4R^2}(ABC) = \frac{R^2 - OP^2}{4R^2}(ABC).$$

Note that $OP^2 - R^2$ is the power σ_P of point P with respect to the circle (ABC). Therefore we finally have

$$(A_1B_1C_1) = -\frac{\sigma_P}{4R^2}(ABC).$$

2°. If point P coincides with point G (the point of intersection of the medians of $\triangle ABC$), then

$$OP^2 = OG^2 = p\bar{p} = \frac{1}{9}(Rz_1 + Rz_2 + Rz_3)(R\bar{z}_1 + R\bar{z}_2 + R\bar{z}_3)$$

$$= \frac{R^2}{9}(3 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_3 + z_3\bar{z}_2 + z_3\bar{z}_1 + z_1\bar{z}_3).$$

But

$$a^2 = (Rz_2 - Rz_3)(R\bar{z}_2 - R\bar{z}_3) = R^2[2 - (z_2\bar{z}_3 + z_3\bar{z}_2)],$$

whence

$$z_2\bar{z}_3 + z_3\bar{z}_2 = 2 - \frac{a^2}{R^2}$$

and, similarly,

$$z_3\bar{z}_1 + z_1\bar{z}_3 = 2 - \frac{b^2}{R^2},$$

$$z_1\bar{z}_2 + z_2\bar{z}_1 = 2 - \frac{c^2}{R^2}$$

so that

$$OG^2 = \frac{R^2}{9}\left(3 + 2 - \frac{a^2}{R^2} + 2 - \frac{b^2}{R^2} + 2 - \frac{c^2}{R^2}\right) = R^2 - \frac{a^2 + b^2 + c^2}{9}.$$

Consequently,

$$R^2 - OG^2 = \frac{a^2 + b^2 + c^2}{9}$$

and therefore

$$(A_1B_1C_1) = \frac{a^2 + b^2 + c^2}{36R^2}(ABC).$$

But

$$R = \frac{abc}{4|(ABC)|},$$

hence

$$(A_1B_1C_1) = \frac{4(a^2 + b^2 + c^2)}{9a^2b^2c^2}(ABC)^3.$$

3°. If point P coincides with the centre I of the circle (I) inscribed in $\triangle\, ABC$, then $p\bar{p} = OP^2 = OI^2 = R^2 - 2Rr$ (*Euler's formula*, see problem 33 below) and, hence,

$$(A_1B_1C_1) = \frac{R^2 - (R^2 - 2Rr)}{4R^2}(ABC) = \frac{r}{2R}(ABC)$$

$$= \frac{2\dfrac{|(ABC)|}{a+b+c}}{\dfrac{abc}{2|(ABC)|}}(ABC) = \frac{4(ABC)^3}{abc(a+b+c)}.$$

Thus,

$$(A_1B_1C_1) = \frac{4(ABC)^3}{abc(a+b+c)}.$$

Remark. Note the formula

$$\frac{(A_1B_1C_1)}{(ABC)} = \frac{r}{2R},$$

where A_1, B_1, C_1 are the projections of point I on the sides BC, CA, AB.

4°. If the point P coincides with the point O, then $p\bar{p} = OP^2 = 0$ and, consequently,

$$(A_1B_1C_1) = \frac{R^2}{4R^2}(ABC) = \frac{1}{4}(ABC)$$

(this is also immediately clear on the basis of elementary-geometry reasoning).

5°. If the point P coincides with the orthocentre H of triangle ABC, we have

$$(A_1B_1C_1) = \frac{R^2 - OH^2}{4R^2}(ABC) = \frac{R^2 - (3OG)^2}{4R^2}(ABC)$$

$$= \frac{R^2 - 9\left(R^2 - \dfrac{a^2+b^2+c^2}{9}\right)}{4R^2}(ABC) = \frac{a^2+b^2+c^2-8R^2}{4R^2}(ABC)$$

$$= \left(\frac{a^2+b^2+c^2}{4R^2} - 2\right)(ABC) = \left(\frac{a^2+b^2+c^2}{\dfrac{a^2b^2c^2}{4(ABC)^2}} - 2\right)(ABC)$$

$$= \left(4\,\frac{a^2+b^2+c^2}{a^2b^2c^2}(ABC)^2 - 2\right)(ABC).$$

Since $\dfrac{a^2}{4R^2} = \sin^2 A$, $\dfrac{b^2}{4R^2} = \sin^2 B$, $\dfrac{c^2}{4R^2} = \sin^2 C$, it follows that

$$\frac{(A_1B_1C_1)}{(ABC)} = \frac{a^2+b^2+c^2}{4R^2} - 2 = \sin^2 A + \sin^2 B + \sin^2 C - 2$$

$$= \frac{1-\cos 2A}{2} + \frac{1-\cos 2B}{2} + \frac{1-\cos 2C}{2} - 2$$

$$= -\frac{1}{2}(1 + \cos 2C + \cos 2A + \cos 2B)$$

$$= -\frac{1}{2}[2\cos^2 C + 2\cos(A+B)\cos(A-B)]$$

$$= -\cos^2 C - \cos(A+B)\cos(A-B) = -\cos^2 C + \cos C\cos(A-B)$$

$$= \cos C[\cos(A-B) - \cos C] = \cos C[\cos(A-B) + \cos(A+B)]$$

$$= 2\cos A\cos B\cos C.$$

Thus, if $A_1B_1C_1$ is a triangle that is orthocentric with respect to $\triangle ABC$ (that is A_1, B_1, C_1 are the feet of the altitudes of $\triangle ABC$), then

$$\frac{(A_1B_1C_1)}{(ABC)} = 2\cos A\cos B\cos C,$$

whence, incidentally, it follows that if triangle ABC is an acute-angled triangle, then $\triangle \overrightarrow{ABC}$ and $\triangle \overrightarrow{A_1B_1C_1}$ have the same orientation (Fig. 9), and if it is an obtuse-angled triangle, then opposite orientation (Fig. 10).

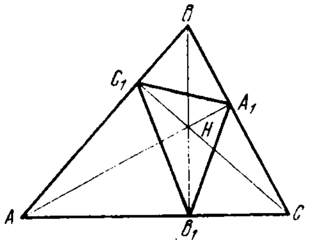

Fig. 9

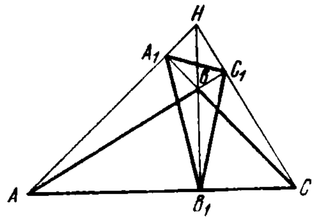

Fig. 10

Problem 7. Given $\triangle ABC$ and a straight line l. Let A_1, B_1, C_1 be the orthogonal projections of points A, B, C on l. Prove that the lines that pass through the points A_1, B_1, C_1 and are respectively perpendicular to the straight lines BC, CA, AB intersect in a single point Ω (called the *orthopole* of the straight line l with respect to the triangle ABC) (Fig. 11). Taking (ABC) for the unit circle and assuming l is given by

$$z - z_0 = \varkappa(\bar{z} - \bar{z}_0), \qquad |\varkappa| = 1, \quad (7)$$

find the affix ω of the orthopole Ω of l with respect to $\triangle ABC$. The affixes of the vertices of $\triangle ABC$ are equal to z_1, z_2, z_3 respectively.

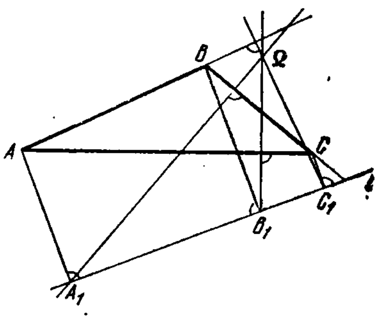

Fig. 11

Solution. The equation of the straight line passing through point A perpendicularly to l is of the form

$$z - z_1 = -\varkappa(\bar{z} - \bar{z}_1). \qquad (8)$$

Rewriting equations (7) and (8) as

$$z - \varkappa\bar{z} = z_0 - \varkappa\bar{z}_0,$$

$$z + \varkappa\bar{z} = z_1 + \varkappa\bar{z}_1$$

and forming the half-sum, we find the affix $z = a_1$ of point A_1:

$$a_1 = \frac{1}{2}(z_0 + z_1 + \varkappa\bar{z}_1 - \varkappa\bar{z}_0).$$

The equation of the straight line passing through point A_1 perpendicularly to BC is of the form

$$z - a_1 = z_2z_3(\bar{z} - \bar{a}_1)$$

or

$$z - \frac{1}{2}(z_0 + z_1 + \varkappa \bar{z}_1 - \varkappa \bar{z}_0) = z_2 z_3 \left[\bar{z} - \frac{1}{2}\left(\bar{z}_0 + \frac{1}{z_1} + \frac{z_1}{\varkappa} - \frac{z_0}{\varkappa}\right)\right]$$

or

$$z - z_2 z_3 \bar{z} = \frac{1}{2}\left(z_0 + z_1 + \frac{\varkappa}{z_1} - \varkappa \bar{z}_0 - \frac{z_2 z_3}{z_1} - z_2 z_3 \bar{z}_0 - \frac{\sigma_3}{\varkappa} + \frac{z_2 z_3 z_0}{\varkappa}\right). \quad (9)$$

Similarly, the equation of the perpendicular dropped from point B_1 on AC is of the form

$$z - z_3 z_1 \bar{z} = \frac{1}{2}\left(z_0 + z_2 + \frac{\varkappa}{z_2} - \varkappa \bar{z}_0 - \frac{z_3 z_1}{z_2} - z_3 z_1 \bar{z}_0 - \frac{\sigma_3}{\varkappa} + \frac{z_3 z_1 z_0}{\varkappa}\right). \quad (10)$$

Subtracting equation (10) from (9) termwise, we find the number $\bar{\omega}$ that is conjugate to the affix ω of the point of intersection of lines (9) and (10):

$$z_3(z_1 - z_2)\,\bar{\omega} = \frac{1}{2}\left[z_1 - z_2 + \varkappa\left(\frac{1}{z_1} - \frac{1}{z_2}\right) + z_3\left(\frac{z_1}{z_2} - \frac{z_2}{z_1}\right) - z_3(z_1 - z_2)\,\bar{z}_0 - \frac{z_3 z_0}{\varkappa}(z_1 - z_2)\right]$$

or, cancelling $z_1 - z_2$,

$$z_3 \bar{\omega} = \frac{1}{2}\left(1 - \frac{\varkappa}{z_1 z_2} + \frac{z_3 z_1 + z_3 z_2}{z_1 z_2} + z_3 \bar{z}_0 - \frac{z_3 z_0}{\varkappa}\right),$$

whence

$$\bar{\omega} = \frac{1}{2}\left(\frac{1}{z_3} - \frac{\varkappa}{z_1 z_2 z_3} + \frac{z_1 + z_2}{z_1 z_2} + \bar{z}_0 - \frac{z_0}{\varkappa}\right)$$

or

$$\bar{\omega} = \frac{1}{2}\left(\frac{\sigma_2}{\sigma_3} - \frac{\varkappa}{\sigma_3} + \bar{z}_0 - \frac{z_0}{\varkappa}\right),$$

whence

$$\omega = \frac{1}{2}\left(\frac{\bar{\sigma}_2}{\bar{\sigma}_3} - \frac{\bar{\varkappa}}{\bar{\sigma}_3} + z_0 - \varkappa \bar{z}_0\right).$$

But $\bar{\sigma}_2 = \frac{\sigma_1}{\sigma_3}$, $\bar{\sigma}_3 = \frac{1}{\sigma_3}$ and so

$$\omega = \frac{1}{2}\left(\sigma_1 - \frac{\sigma_3}{\varkappa} + z_0 - \varkappa \bar{z}_0\right). \quad (11)$$

The symmetry of this equation with respect to z_1, z_2, z_3 permits stating that the perpendicular dropped from point C_1 on line AB will also pass

through the point with affix ω defined by (11), that is, (11) defines the affix of the orthopole of the straight line l with respect to $\triangle ABC$. Incidentally, this is evident from the following: write down the equation of the perpendicular dropped from C_1 on the line AB and convince yourself that the number ω given by (11) satisfies the equation of that perpendicular.

In particular, if l passes through the centre O of the unit circle $(O) = (ABC)$, then the affix ω of its orthopole with respect to $\triangle ABC$ is

$$\omega = \frac{1}{2}\left(\sigma_1 - \frac{\sigma_3}{\varkappa}\right), \tag{12}$$

and if the Boutain point is taken for the unit point, then

$$\omega = \frac{1}{2}(\sigma_1 - \bar{\varkappa}). \tag{13}$$

Figure 11 depicts the construction of the orthopole Ω of line l with respect to $\triangle ABC$.

Remark. There is an elegant synthetic proof (which belongs to the French geometer *R*. Deaux) that the straight lines l_1, l_2, l_3 that pass through the points A_1, B_1, C_1 and are perpendicular to the lines BC, CA, AB respectively intersect in one point Ω. Let A_2, B_2, C_2 be the midpoints of sides BC, CA, AB, then the radical axes of the circles (A_2, A_2C_1), (B_2, B_2A_1), (C_2, C_2B_1) taken pairwise are the straight lines l_1, l_2, l_3, and therefore they intersect in the single point Ω (Fig. 12).

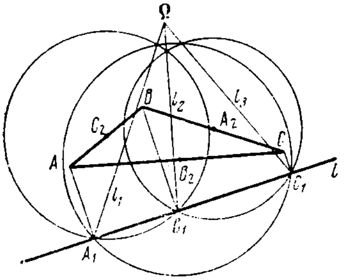

Fig. 12

Problem 8. Given a triangle ABC and a straight line l. The circle $(O) = (ABC)$ is taken to be the unit circle; z_1, z_2, z_3 are the affixes of the points A, B, C, respectively. The line l is given by the self-conjugate equation

$$\bar{a}z + a\bar{z} = b \tag{14}$$

where $a \neq 0$ and b is a real number. Find the affix ω of the orthopole Ω of line l with respect to the triangle ABC (see the preceding problem).

Solution. The slope of the given straight line is equal to $\varkappa = -\dfrac{a}{\bar{a}}$ and the slope of the line perpendicular to l is equal to $\varkappa' = \dfrac{a}{\bar{a}}$.

The equation of the straight line that passes through point A perpendicularly to l is of the form

$$z - z_1 = \frac{a}{\bar{a}}(\bar{z} - \bar{z}_1)$$

or

$$\bar{a}z - a\bar{z} = \bar{a}z_1 - a\bar{z}_1. \tag{15}$$

Adding the equations (14) and (15) termwise, we find the affix a_1 of the projection A_1 of point A on l:

$$a_1 = \frac{\bar{a}z_1 - a\bar{z}_1 + b}{2\bar{a}}.$$

The equation of the straight line passing through point A_1 perpendicularly to BC is of the form

$$z - a_1 = z_2z_3(\bar{z} - \bar{a}_1)$$

or

$$z - \frac{\bar{a}z_1 - a\bar{z}_1 + b}{2\bar{a}} = z_2z_3\left(\bar{z} - \frac{a\bar{z}_1 - \bar{a}z_1 + b}{2a}\right)$$

or

$$z - z_2z_3\bar{z} = \frac{\bar{a}z_1 - a\bar{z}_1 + b}{2\bar{a}} - z_2z_3\frac{a\bar{z}_1 - \bar{a}z_1 + b}{2a}. \tag{16}$$

In similar fashion we can write down the equation of the perpendicula to AC that passes through the projection B_1 of point B onto l:

$$z - z_3z_1\bar{z} = \frac{\bar{a}z_2 - a\bar{z}_2 + b}{2\bar{a}} - z_3z_1\frac{a\bar{z}_2 - \bar{a}z_2 + b}{2a}. \tag{17}$$

Subtracting equation (17) from (16) term by term, we get the number $\bar{\omega}$, which is the conjugate of the affix ω of the orthopole Ω of line l with respect to triangle ABC:

$$(z_1 - z_2)\,z_3\bar{\omega} = \frac{(z_1 - z_2)\,\bar{a} + \left(\dfrac{1}{z_2} - \dfrac{1}{z_1}\right)a}{2\bar{a}} + \frac{z_2\left(\dfrac{z_1}{z_2} - \dfrac{z_2}{z_1}\right)a + bz_3(z_1 - z_2)}{2a},$$

and, cancelling $z_1 - z_2$, we obtain

$$z_3\bar{\omega} = \frac{\bar{a} + \dfrac{a}{z_1z_2}}{2\bar{a}} + \frac{\dfrac{z_3(z_1 + z_2)}{z_1z_2}a + bz_3}{2a},$$

whence

$$\bar{\omega} = \frac{1}{2z_3} + \frac{a}{2\bar{a}\sigma_3} + \frac{1}{2z_1} + \frac{1}{2z_2} + \frac{b}{2a}$$

or

$$\bar{\omega} = \frac{1}{2}\bar{\sigma}_1 + \frac{a\bar{\sigma}_3}{2\bar{a}} + \frac{b}{2a}$$

and, hence,

$$\omega = \frac{1}{2}\left(\sigma_1 + \frac{\bar{a}\sigma_3}{a} + \frac{b}{\bar{a}}\right). \tag{18}$$

If l passes through the centre of the unit circle, then $b = 0$ and formula (18) becomes

$$\omega = \frac{1}{2}\left(\sigma_1 + \frac{\bar{a}}{a}\sigma_3\right), \tag{19}$$

and if the unit point is a Boutain point ($\sigma_3 = 1$), then

$$\omega = \frac{1}{2}\left(\sigma_1 + \frac{\bar{a}}{a}\right). \tag{20}$$

The equations (19) and (20) coincide, respectively, with (12) and (13) of the preceding problem, since $\frac{\bar{a}}{a} = -\bar{\varkappa}$, where $\varkappa = -\frac{a}{\bar{a}}$ is the slope of l given by the equation $\bar{a}z + a\bar{z} = b$.

Problem 9. Let A_1, B_1, C_1 be the feet of the altitudes of $\triangle ABC$ inscribed in the circle $(ABC) = (O)$, which we take to be the unit circle. Points P, Q, R are chosen on the straight lines AA_1, BB_1, CC_1 so that

$$\frac{\overrightarrow{AA_1}}{\overrightarrow{AP}} = \frac{\overrightarrow{BB_1}}{\overrightarrow{BQ}} = \frac{\overrightarrow{CC_1}}{\overrightarrow{CR}} = \lambda.$$

Find the ratio

$$\mu = \frac{(PQR)}{(ABC)}.$$

Express this ratio in terms of the interior angles A, B, C of the given $\triangle ABC$ and in terms of λ. Consider the special cases: (1) $\lambda = 1$, (2) $\lambda = -1$, (3) $\lambda = 1/2$, (4) $\lambda = -1/2$, (5) $\lambda = 2$, (6) $\lambda = -2$.

Solution. Take the circle (O) as the unit circle. Let z_1, z_2, z_3 be the affixes of the points A, B, C, respectively. The equations of the straight lines BC and AA_1 are

$$z + z_2z_3\bar{z} = z_2 + z_3,$$

$$z - z_2z_3\bar{z} = z_1 - \frac{z_2z_3}{z_1}.$$

Combining these equations term by term, we find the affix $z = a_1$ of point A_1:

$$a_1 = \frac{1}{2}\left(\sigma_1 - \frac{z_2z_3}{z_1}\right).$$

and then from the relation

$$\frac{\overrightarrow{AA_1}}{\overrightarrow{AP}} = \lambda$$

we find the affix p of point P:

$$\frac{a_1 - z_1}{p - z_1} = \lambda, \qquad a_1 - z_1 = \lambda p - \lambda z_1,$$

$$\lambda p = a_1 - z_1 + \lambda z_1 = \frac{1}{2}\left(\sigma_1 - \frac{z_2z_3}{z_1}\right) - z_1 + \lambda z_1,$$

whence

$$p = \frac{1}{2\lambda}\left(\sigma_1 - \frac{\sigma_3}{z_1^2}\right) + \frac{\lambda - 1}{\lambda} z_1.$$

Similarly,

$$q = \frac{1}{2\lambda}\left(\sigma_1 - \frac{\sigma_3}{z_2^2}\right) + \frac{\lambda - 1}{\lambda} z_2,$$

$$r = \frac{1}{2\lambda}\left(\sigma_1 - \frac{\sigma_3}{z_3^2}\right) + \frac{\lambda - 1}{\lambda} z_3,$$

where q and r are the affixes of points Q and R. We now find

$$(PQR) = \frac{i}{4}\begin{vmatrix} -\frac{\sigma_3}{2\lambda z_1^2} + \frac{\lambda-1}{\lambda} z_1 + \frac{\sigma_1}{2\lambda} & -\frac{z_1^2}{2\lambda\sigma_3} + \frac{\lambda-1}{\lambda}\frac{1}{z_1} + \frac{\sigma_2}{2\lambda\sigma_3} & 1 \\ -\frac{\sigma_3}{2\lambda z_2^2} + \frac{\lambda-1}{\lambda} z_2 + \frac{\sigma_1}{2\lambda} & -\frac{z_2^2}{2\lambda\sigma_3} + \frac{\lambda-1}{\lambda}\frac{1}{z_2} + \frac{\sigma_2}{2\lambda\sigma_3} & 1 \\ -\frac{\sigma_3}{2\lambda z_3^2} + \frac{\lambda-1}{\lambda} z_3 + \frac{\sigma_1}{2\lambda} & -\frac{z_3^2}{2\lambda\sigma_3} + \frac{\lambda-1}{\lambda}\frac{1}{z_3} + \frac{\sigma_2}{2\lambda\sigma_3} & 1 \end{vmatrix} =$$

$$= \frac{i}{4}\begin{vmatrix} -\dfrac{\sigma_3}{2\lambda z_1^2}+\dfrac{\lambda-1}{\lambda}z_1 & -\dfrac{z_1^2}{2\lambda\sigma_3}+\dfrac{\lambda-1}{\lambda}\dfrac{1}{z_1} & 1 \\ -\dfrac{\sigma_3}{2\lambda z_2^2}+\dfrac{\lambda-1}{\lambda}z_2 & -\dfrac{z_2^2}{2\lambda\sigma_3}+\dfrac{\lambda-1}{\lambda}\dfrac{1}{z_2} & 1 \\ -\dfrac{\sigma_3}{2\lambda z_3^2}+\dfrac{\lambda-1}{\lambda}z_3 & -\dfrac{z_3^2}{2\lambda\sigma_3}+\dfrac{\lambda-1}{\lambda}\dfrac{1}{z_3} & 1 \end{vmatrix}$$

$$= \frac{i}{4}\left(\frac{1}{4\lambda^2}\begin{vmatrix} z^{-2} & z_1^2 & 1 \\ z_2^{-2} & z_2^2 & 1 \\ z_3^{-2} & z_3^2 & 1 \end{vmatrix} - \frac{\sigma_3(\lambda-1)}{2\lambda^2}\begin{vmatrix} z_1^{-2} & z_1^{-1} & 1 \\ z_2^{-2} & z_2^{-1} & 1 \\ z_3^{-2} & z_3^{-1} & 1 \end{vmatrix}\right.$$

$$\left. - \frac{\lambda-1}{2\lambda^2\sigma_3}\begin{vmatrix} z_1 & z_1^2 & 1 \\ z_2 & z_2^2 & 1 \\ z_3 & z_3^2 & 1 \end{vmatrix} + \frac{(\lambda-1)^2}{\lambda^2}\begin{vmatrix} z_1 & z_1^{-1} & 1 \\ z_2 & z_2^{-1} & 1 \\ z_3 & z_3^{-1} & 1 \end{vmatrix}\right). \tag{21}$$

Furthermore,

$$\begin{vmatrix} z_1^{-2} & z_1^2 & 1 \\ z_2^{-2} & z_2^2 & 1 \\ z_3^{-2} & z_3^2 & 1 \end{vmatrix} = -\frac{1}{\sigma_3^2}\begin{vmatrix} 1 & z_1^2 & z_1^4 \\ 1 & z_2^2 & z_2^4 \\ 1 & z_3^2 & z_3^4 \end{vmatrix} = -\frac{1}{\sigma_3^2}(z_2^2-z_1^2)(z_3^2-z_2^2)(z_3^2-z_1^2)$$

$$= -\frac{1}{\sigma_3^2}(z_2-z_1)(z_3-z_2)(z_3-z_1)(z_2+z_1)(z_3+z_2)(z_3+z_1)$$

$$= -\frac{1}{\sigma_3^2}\begin{vmatrix} 1 & z_1 & z_1^2 \\ 1 & z_2 & z_2^2 \\ 1 & z_3 & z_3^2 \end{vmatrix}(z_2+z_1)\,(z_3+z_2)\,(z_3+z_1)$$

$$= -\frac{1}{\sigma_3}\begin{vmatrix} \bar{z}_1 & 1 & z_1 \\ \bar{z}_2 & 1 & z_2 \\ \bar{z}_3 & 1 & z_3 \end{vmatrix}(\sigma_1-z_1)(\sigma_1-z_2)(\sigma_1-z_3)$$

$$= -\frac{1}{\sigma_3}\frac{4}{i}(ABC)\,(\sigma_1\,\sigma_2-\sigma_3) = -\frac{4}{i}(ABC)(\sigma_1\bar{\sigma}_1-1),$$

and the first term in round brackets in (21) is

$$-\frac{i(1-\sigma\bar{\sigma}_1)}{\lambda^2}(ABC). \tag{22}$$

Subtracting the second term, we get

$$\begin{vmatrix} z_1^{-2} & z_1^{-1} & 1 \\ z_2^{-2} & z_2^{-1} & 1 \\ z_3^{-2} & z_3^{-1} & 1 \end{vmatrix} = \frac{1}{\sigma^3} \begin{vmatrix} \bar{z}_1 & 1 & z_1 \\ \bar{z}_2 & 1 & z_2 \\ \bar{z}_3 & 1 & z_3 \end{vmatrix} = -\frac{4i}{\sigma_3}(ABC),$$

so that the second term in round brackets in (21) is equal to

$$\frac{2i(\lambda - 1)}{\lambda^2}(ABC). \tag{23}$$

Furthermore,

$$\begin{vmatrix} z_1 & z_1^2 & 1 \\ z_2 & z_2^2 & 1 \\ z_3 & z_3^2 & 1 \end{vmatrix} = \sigma_3 \begin{vmatrix} 1 & z_1 & \bar{z}_1 \\ 1 & z_2 & \bar{z}_2 \\ 1 & z_3 & \bar{z}_3 \end{vmatrix} = -4i\sigma_3(ABC),$$

and so the third term in round brackets in (21) is

$$\frac{2i(\lambda - 1)}{\lambda^2}(ABC). \tag{24}$$

Finally, the last term is equal to

$$-4i\frac{(\lambda - 1)^2}{\lambda^2}(ABC). \tag{25}$$

From the formulas (21)-(25) it follows that

$$\begin{aligned}(PQR) &= \frac{i}{4}\left[-\frac{i(1-\sigma_1\bar{\sigma}_1)}{\lambda^2} + \frac{2i(\lambda-1)}{\lambda^2}\right. \\ &\quad \left. + \frac{2i(\lambda-1)}{\lambda^2} - 4i\frac{(\lambda-1)^2}{\lambda^2}\right](ABC) \\ &= \left[\frac{1-\sigma_1\bar{\sigma}_1}{4\lambda^2} - \frac{\lambda-1}{\lambda^2} + \frac{(\lambda-1)^2}{\lambda^2}\right](ABC) \\ &= \left[\frac{1-\sigma_1\bar{\sigma}_1}{4\lambda^2} + \left(1-\frac{1}{\lambda}\right)\left(1-\frac{2}{\lambda}\right)\right](ABC),\end{aligned}$$

whence

$$\begin{aligned}\frac{(PQR)}{(ABC)} &= \frac{1-\sigma_1\bar{\sigma}_1}{4\lambda^2} + \left(1-\frac{1}{\lambda}\right)\left(1-\frac{2}{\lambda}\right) \\ &= -\frac{\sigma_1\bar{\sigma}_1 - 1}{4\lambda^2} + \left(1-\frac{1}{\lambda}\right)\left(1-\frac{2}{\lambda}\right).\end{aligned} \tag{26}$$

But $\sigma\bar{\sigma}_1 - 1$ is the power of the orthocentre H of $\triangle ABC$ with respect to the circle (ABC). We have

$$\sigma\bar{\sigma}_1 - 1 = (z_1 + z_2 + z_3)(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) - 1$$
$$= 2 + (z_1\bar{z}_2 + z_2\bar{z}_1) + (z_2\bar{z}_3 + z_3\bar{z}_2) + (z_3\bar{z}_1 + z_1\bar{z}_3).$$

Furthermore,

$$AB^2 = c^2 = (z_2 - z_1)(\bar{z}_2 - \bar{z}_1) = 2 - (z_1\bar{z}_2 + z_2\bar{z}_1),$$

whence

$$z_1\bar{z}_2 + z_2\bar{z}_1 = 2 - c^2 = 2 - 4\sin^2 C,$$

since

$$\frac{c}{\sin C} = 2R = 2.$$

In similar fashion we find

$$z_2\bar{z}_3 + z_3\bar{z}_2 = 2 - 4\sin^2 A, \quad z_3\bar{z}_1 + z_1\bar{z}_3 = 2 - 4\sin^2 B$$

and therefore (see problem 6)

$$\sigma_1\bar{\sigma}_1 - 1 = 4(2 - (\sin^2 A + \sin^2 B + \sin^2 C)) = -8\cos A\cos B\cos C,$$

and formula (26) becomes

$$\mu = \frac{(PQR)}{(ABC)} = \frac{2\cos A\cos B\cos C}{\lambda^2} + \left(1 - \frac{1}{\lambda}\right)\left(1 - \frac{2}{\lambda}\right). \quad (27)$$

(1) If $\lambda = 1$, then the points P, Q, R coincide, respectively, with the feet of the altitudes of triangle ABC.

Answer. $\mu = 2\cos A\cos B\cos C$.

(2) If $\lambda = -1$, then the points P, Q, R are symmetric to the feet A_1, B_1, C_1 of the altitudes of $\triangle ABC$ with respect to its vertices A, B, C.

Answer. $\mu = 6 + 2\cos A\cos B\cos C$.

(3) If $\lambda = 1/2$, then the points P, Q, R are symmetric to the vertices A, B, C of $\triangle ABC$ with respect to its sides.

Answer. $\mu = 3 + 8\cos A\cos B\cos C$.

(4) If $\lambda = -1/2$, then the points P, Q, R are obtained from the points A_1, B_1, C_1 via the homothetic transformations $(A, -2)$, $(B, -2)$, $(C, -2)$.

Answer. $\mu = 15 + 8\cos A\cos B\cos C$.

(5) If $\lambda = 2$, then P, Q, R are the midpoints of the altitudes AA_1, BB_1, CC_1 of $\triangle ABC$.

Answer. $\mu = \frac{1}{2}\cos A\cos B\cos C$.

(6) If $\lambda = -2$, then the points P, Q, R are symmetric to the midpoints A_2, B_2, C_2 of the altitudes AA_1, BB_1, CC_1 of $\triangle ABC$ with respect to its vertices.

Answer. $\mu = 3 + \frac{1}{2}\cos A\cos B\cos C$.

Problem 10. Let A', B', C' be points symmetric to the vertices A, B, C of $\triangle ABC$ with respect to its sides BC, CA, AB. What relationship between the angles A, B, C of $\triangle ABC$ is necessary and sufficient for the straight lines AB', BC', CA' to pass through one point (Fig. 13)?

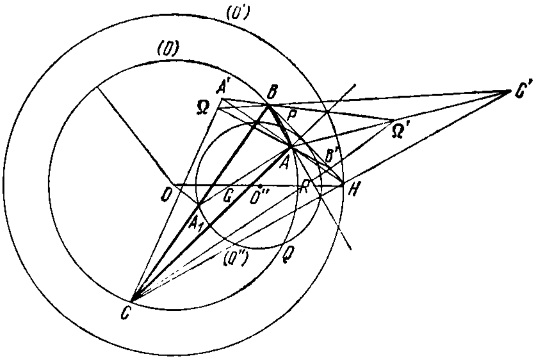

Fig. 13

Solution. Take (ABC) for the unit circle.

We find the affix b' of point B'. The equation of the straight line AC is

$$z + z_3 z_1 \bar{z} = z_1 + z_3.$$

The equation of the straight line passing through point B perpendicularly to AC is

$$z - z_2 = z_3 z_1 (\bar{z} - \bar{z}_2)$$

or

$$z - z_3 z_1 \bar{z} = z_2 - \frac{z_3 z_1}{z_2}.$$

From the equation of AC and from this equation we find the affix of the projection of point B on line AC:

$$z = \frac{1}{2}\left(z_1 + z_2 + z_3 - \frac{z_3 z_1}{z_2}\right).$$

The affix b' of point B', which is symmetric to point B with respect to AC, is found from the relation

$$\frac{b' + z_2}{2} = \frac{1}{2}\left(z_1 + z_2 + z_3 - \frac{z_3 z_1}{z_2}\right),$$

whence

$$b' = z_1 + z_3 - \frac{z_3 z_1}{z_2}.$$

From this we have

$$\bar{b}' = \bar{z}_1 + \bar{z}_3 - \frac{z_2}{z_3 z_1} = \frac{z_1 + z_3 - z_2}{z_1 z_3}.$$

We now set up an equation of the straight line AB':

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ b' & \bar{b}_1' & 1 \end{vmatrix} = 0$$

or

$$z(\bar{z}_1 - \bar{b}') + (b' - z_1)\,\bar{z} + z_1\bar{b}' - b'\bar{z}_1 = 0.$$

We have

$$\bar{z}_1 - \bar{b}' = \frac{1}{z_1} - \frac{z_3 + z_1 - z_2}{z_3 z_1} = \frac{z_2 - z_1}{z_3 z_1},$$

$$z_1 - b' = \frac{\dfrac{1}{z_2} - \dfrac{1}{z_1}}{\dfrac{1}{z_3 z_1}} = \frac{z_3(z_1 - z_2)}{z_2}, \quad b' - z_1 = \frac{z_3(z_2 - z_1)}{z_2},$$

$$z_1\bar{b}' - b'\bar{z}_1 = z_1 \frac{z_3 + z_1 - z_2}{z_3 z_1} - \frac{1}{z_1}\left(z_3 + z_1 - \frac{z_3 z_1}{z_2}\right)$$

$$= \frac{z_3 + z_1 - z_2}{z_3} - \frac{z_3}{z_1} - 1 + \frac{z_3}{z_2} = \frac{z_1}{z_3} - \frac{z_3}{z_1} + \frac{z_3}{z_2} - \frac{z_2}{z_3}$$

$$= \frac{1}{z_3}(z_1 - z_2) + z_3\left(\frac{1}{z_2} - \frac{1}{z_1}\right) = \frac{1}{z_3}(z_1 - z_2) + z_3 \frac{z_1 - z_2}{z_1 z_2}$$

$$= (z_1 - z_2)\left(\frac{1}{z_3} + \frac{z_3}{z_1 z_2}\right),$$

and the equation of AB' takes the form

$$\frac{1}{z_3 z_1} z + \frac{z_3}{z_2}\bar{z} - \frac{1}{z_3} - \frac{z_3}{z_1 z_2} = 0$$

or

$$z_2 z + z_3^2 z_1 \bar{z} - z_3^2 - z_1 z_2 = 0.$$

In similar fashion we can write down the equations of the straight lines BC' and CA'. Thus,

$$z_2 z + z_3^2 z_1 \bar{z} - z_3^2 - z_1 z_2 = 0, \quad (AB')$$

$$z_3 z + z_1^2 z_2 \bar{z} - z_1^2 - z_2 z_3 = 0, \quad (BC')$$

$$z_1 z + z_2^2 z_3 \bar{z} - z_2^2 - z_3 z_1 = 0. \quad (CA')$$

Denote by K, L, M the points of intersection of the lines BC' and CA', CA' and AB', AB' and BC'. Then

$$(KLM) = \frac{i}{4} \frac{\begin{vmatrix} z_2 & z_3^2 z_1 & z_3^2 + z_1 z_2 \\ z_3 & z_1^2 z_2 & z_1^2 + z_2 z_3 \\ z_1 & z_2^2 z_3 & z_2^2 + z_3 z_1 \end{vmatrix}^2}{\sigma_3 (z_2 z_3^2 - z_1^3)(z_3 z_1^2 - z_2^3)(z_1 z_2^2 - z_3^3)}.$$

Furthermore,

$$\Delta = \begin{vmatrix} z_2 & z_3^2 z_1 & z_3^2 + z_1 z_2 \\ z_3 & z_1^2 z_2 & z_1^2 + z_2 z_3 \\ z_1 & z_2^2 z_3 & z_2^2 + z_3 z_1 \end{vmatrix} = z_1^2 z_2^4 + z_1^3 z_2^3 z_3 + z_2^2 z_3^4 + z_1 z_3^2 z_2^3 + z_1^4 z_3^2 + z_1^2 z_2 z_3^3$$

$$-z_1^3 z_2 z_3^2 - z_1^4 z_2^2 - z_1^2 z_2^3 z_3 - z_2^4 z_3^2 - z_1 z_2^2 z_3^3 - z_1^2 z_3^4$$

$$= z_1^2 z_2^2 (z_2^2 - z_1^2) - z_1^2 z_2^2 z_3 (z_2 - z_1) + z_3^4 (z_2^2 - z_1^2) + z_3^2 z_2 z_1 (z_2^2 - z_1^2) - z_3^2 (z_2^4 - z_1^4)$$

$$- z_3^3 z_1 z_2 (z_2 - z_1) = (z_2 - z_1)(z_1^2 z_2^3 + z_1^3 z_2^2 - z_1^2 z_2^2 z_3 + z_3^4 z_2 + z_3^4 z_1 + z_3^2 z_2^2 z_1$$

$$+ z_3^2 z_2 z_1^2 - z_3^2 z_2^3 - z_3^2 z_2^2 z_1 - z_3^2 z_2 z_1^2 - z_3^2 z_1^3 - z_3^3 z_1 z_2) = (z_2 - z_1)[z_1^2 z_2^2 (z_1 - z_3)$$

$$+ z_2^3 (z_1^2 - z_3^2) - z_2 z_3^2 (z_1^2 - z_3^2) - z_1 z_3^2 (z_1^2 - z_3^2) + z_3^2 z_2 z_1 (z_1 - z_3)]$$

$$= (z_2 - z_1)(z_1 - z_3)(z_1^2 z_2^2 + z_2^3 z_1 + z_2^3 z_3 - z_2 z_1 z_3^2 - z_2 z_3^3 - z_1^2 z_3^2 - z_1 z_3^3 + z_3^2 z_1 z_2)$$

$$= (z_2 - z_1)(z_1 - z_3)[z_1^2 (z_2^2 - z_3^2) + z_1 (z_2^3 - z_3^3) + z_2 z_3 (z_2^2 - z_3^2)]$$

$$(z_2 - z_1)(z_1 - z_3)(z_2 - z_3)(z_1^2 z_2 + z_1^2 z_3 + z_1 z_2^2 + z_1 z_2 z_3 + z_1 z_3^2 + z_2^2 z_3 + z_2 z_3^2)$$

$$(z_2 - z_1)(z_3 - z_1)(z_3 - z_2)(z_1^2 z_2 + z_2^2 z_1 + z_2^2 z_3 + z_3^2 z_2 + z_3^2 z_1 + z_1^2 z_3 + z_1 z_2 z_3).$$

Consider the product

$$(z_2 + z_3)(z_3 + z_1)(z_1 + z_2) = \sigma_1 \sigma_2 - \sigma_3$$

$$= 2 z_1 z_2 z_3 + z_1^2 z_2 + z_2^2 z_1 + z_2^2 z_3 + z_3^2 z_2 + z_3^2 z_1 + z_1^2 z_3.$$

From this it follows that the last factor $\sigma_1 \sigma_2 - \sigma_3 - \sigma_3 = \sigma_1 \sigma_2 - 2\sigma_3$ so that

$$\Delta = (z_2 - z_1)(z_3 - z_1)(z_3 - z_2)(\sigma_1 \sigma_2 - 2\sigma_3).$$

Thus,

$$(KLM) = \frac{i}{4}\,\frac{(z_2 - z_1)^2(z_3 - z_1)^2(z_3 - z_2)^2(\sigma_1\sigma_2 - 2\sigma_3)^2}{\sigma_3(z_2z_3^2 - z_1^3)(z_3z_1^2 - z_2^3)(z_1z_2^2 - z_3^3)}.$$

Note that

$$(z_3 - z_2)(z_3 - z_1)(z_2 - z_1)$$

$$= \begin{vmatrix} 1 & z_1 & z_1^2 \\ 1 & z_2 & z_2^2 \\ 1 & z_3 & z_3^2 \end{vmatrix} = \sigma_3 \begin{vmatrix} \bar{z}_1 & 1 & z_1 \\ \bar{z}_2 & 1 & z_2 \\ \bar{z}_3 & 1 & z_3 \end{vmatrix} = \sigma_3 \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = -4i\sigma_3(ABC),$$

so that

$$(KLM) = -\frac{4i\sigma_3(\sigma_1\sigma_2 - 2\sigma_3)^2(ABC)^2}{(z_2z_3^2 - z_1^3)(z_3z_1^2 - z_2^3)(z_1z_2^2 - z_3^3)}.$$

The lines AB', BC', CA' pass through one point if and only if

$$\sigma_1\sigma_2 - 2\sigma_3 = 0$$

or

$$\sigma_1\frac{\sigma_2}{\sigma_3} - 2 = 0, \quad \sigma_1\bar{\sigma}_1 - 2 = 0,$$

that is,

$$\sigma_1\bar{\sigma}_1 = OH^2 = 2 = 2R^2 \quad (R = 1)$$

or

$$OH = R\sqrt{2}.$$

Incidentally, it follows from this that the triangle ABC is an obtuse-angled triangle since $OH > R$, that is, the orthocentre H lies outside the circle (ABC).

A triangle for which $OH = R\sqrt{2}$ may be constructed as follows: construct two concentric circles (O) and (O') with radii 1 and $\sqrt{2}$ (see Fig. 13). On (O') take an arbitrary point H and on (O) an arbitrary point A. Divide the line segment $\overrightarrow{OH}$ in the ratio $1:2$:

$$\overrightarrow{OG} : \overrightarrow{GH} = 1:2.$$

Then G is the point of intersection of the medians of the triangle ABC. Join A and G and on the extension of segment AG beyond point G lay off segment $GA_1 = AG/2$. The point A_1 is the midpoint of side BC and therefore, by drawing a line through A_1 perpendicular to the straight line OA_1 we obtain points B and C as points of intersection with the circle (O).

The condition that makes the construction possible is that point A_1 lie inside circle (O). Since point A_1 is the image of point A under the homothetic transformation $\left(G, -\frac{1}{2}\right)$, and under that homothetic transformation the circle (O) goes into the circle (O''), the radius of which is equal to $\frac{1}{2}$, and the centre O'' lies on segment OH, $OO''=\frac{\sqrt{2}}{3}+\frac{\sqrt{2}}{6}=\frac{\sqrt{2}}{2}$, it follows that the circles (O) and (O'') intersect in points P and Q. The point A can therefore describe an open arc PRQ of the circle (O). From the triangle OPO'' we have

$$PO''^2 = OP^2 + OO''^2 - 2OP \cdot OO'' \cos(\alpha/2),$$

where $\alpha = \angle POQ$. And since $PO'' = 1/2$, $OO'' = \sqrt{2}/2$, $OP = 1$, it follows that

$$\frac{1}{4} = 1 + \frac{1}{2} - 2 \cdot 1 \cdot \frac{\sqrt{2}}{2} \cos \frac{\alpha}{2},$$

whence

$$\cos \frac{\alpha}{2} = \frac{5}{4\sqrt{2}}$$

and, consequently,

$$\cos \alpha = 2 \cdot \frac{25}{32} - 1 = \frac{9}{16}, \quad \alpha = \angle POQ = \arccos (9/16).$$

The condition $OH^2 = 2R^2$ may be rewritten in a different form. Since $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$ (see problem 6), it follows that

$$9R^2 - (a^2 + b^2 + c^2) = 2R^2$$

or

$$a^2 + b^2 + c^2 = 7R^2.$$

And, in yet another form (since $a = 2R \sin A$ and so forth),

$$\sin^2 A + \sin^2 B + \sin^2 C = 7/4$$

or

$$\cos A \cos B \cos C = -1/8$$

(from this it also follows that ABC is an obtuse-angled triangle).

Problem 11. Inscribed in a circle (O) is a regular 14-gon:

$$A_1A_2A_3A_4A_5A_6A_7A_8A_9A_{10}A_{11}A_{12}A_{13}A_{14}.$$

Let us consider the triangle $T = ABC$ with vertices

$$A = A_7, \quad B = A_1, \quad C = A_3.$$

The angles of this triangle intercept arcs $\overset{\frown}{A_1A_3}$, $\overset{\frown}{A_3A_7}$, $\overset{\frown}{A_7A_1}$ and, hence, $A = \pi/7$, $B = 2\pi/7$, $C = 4\pi/7$ (Fig. 14) (that is to say, the angles of tri-

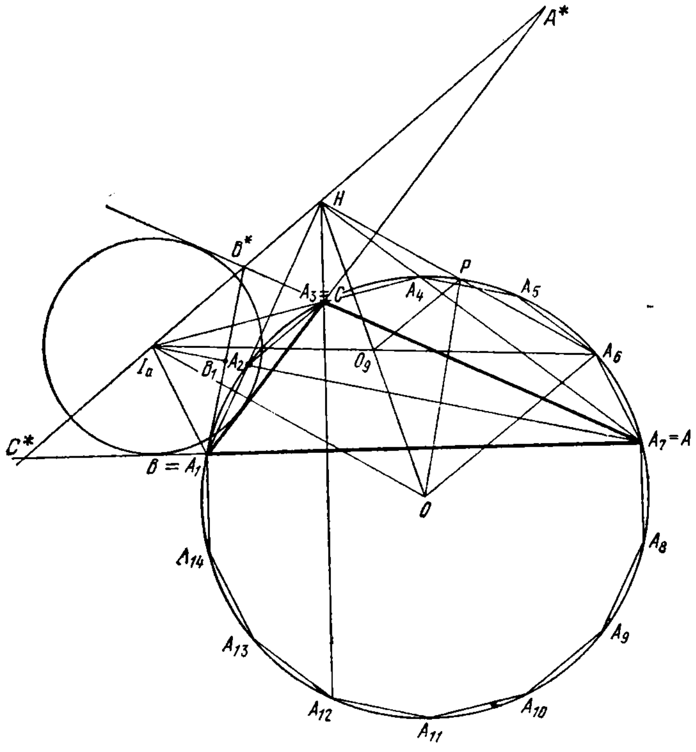

Fig. 14

angle T form a geometric progression with ratio 2). Denote by (I_a) the circle escribed in the angle A of $\triangle ABC$, and by H the orthocentre of $\triangle ABC$.

Prove that:

1°. $OH = OI_a = R\sqrt{2}$ (R is the radius of (O)).

2°. $R = 2r_a$ (r_a is the radius of circle (I_a)).

3°. $I_aH = R$.

4°. $a^2 + b^2 + c^2 = 7R^2$ (here, a, b, c are the lengths of sides BC, CA, AB of the triangle; do not confuse with the affixes of its vertices!).

5°. OI_aHA_6 is a parallelogram whose centre coincides with the centre of the Euler circle of triangle ABC.

6°. The midpoint P of segment HA_6 coincides with one of the points of intersection of (O) and (O_9) ((O_9) is the Euler circle of $\triangle ABC$).

7°. The triangles $\overrightarrow{AI_aH}$, $\overrightarrow{HBI_a}$, $\overrightarrow{I_aHC}$ are similar. Determine their orientation (which pairs have the same orientation and which are oppositely oriented).

8°. The straight lines BC, CA and AB intersect line HI_a in points symmetric to the points A, B, C with respect to the bisectors of the angles C, A, B of triangle ABC (for angles C and B take the bisectors of the exterior angles).

9°. The squares of the lengths of the sides of $\triangle OI_aA_6$ and the squares of the lengths of the sides of $\triangle A_6I_aH$ form a geometric progression with ratio 2.

Solution. 1°. Take (O) as the unit circle. Give point A_1 the affix 1 (that s, A_1 is the unit point). Then the affixes a_k of points A_k are

$$a_k = \cos\frac{(k-1)\pi}{7} + i\sin\frac{(k-1)\pi}{7},$$

$$k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14.$$

The affixes of the vertices $A = A_7$, $B = A_1$, $C = A_3$ are

$$a = a_7 = \cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7},$$

$$b = a_1 = 1,$$

$$c = a_3 = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}.$$

Their sum is equal to the affix h of orthocentre H of $\triangle ABC$:

$$h = 1\cos\frac{2\pi}{7} + \cos\frac{6\pi}{7} + i\left(\sin\frac{2\pi}{7} + \sin\frac{6\pi}{7}\right),$$

whence

$$OH^2 = \left(1 + \cos\frac{2\pi}{7} + \cos\frac{6\pi}{7}\right)^2 + \left(\sin\frac{2\pi}{7} + \sin\frac{6\pi}{7}\right)^2$$

$$= 1 + \cos^2\frac{2\pi}{7} + \cos^2\frac{6\pi}{7} + 2\cos\frac{2\pi}{7} + 2\cos\frac{6\pi}{7} + 2\cos\frac{2\pi}{7}\cos\frac{6\pi}{7}$$

$$+\sin^2\frac{2\pi}{7} + \sin^2\frac{6\pi}{7} + 2\sin\frac{2\pi}{7}\sin\frac{6\pi}{7}$$

$$= 3 + 2\left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right).$$

Let

$$x = 2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7};$$

then

$$x\sin\frac{\pi}{7} = 2\sin\frac{\pi}{7}\cos\frac{2\pi}{7} + 2\sin\frac{\pi}{7}\cos\frac{4\pi}{7} + 2\sin\frac{\pi}{7}\cos\frac{6\pi}{7}$$

$$= -\sin\frac{\pi}{7} + \sin\frac{3\pi}{7} - \sin\frac{3\pi}{7} + \sin\frac{5\pi}{7} - \sin\frac{5\pi}{7} + \sin\pi = -\sin\frac{\pi}{7}$$

and, consequently, $x = -1$; therefore

$$OH^2 = 3 - 1 = 2,$$

whence

$$OH = \sqrt{2} = R\sqrt{2} \quad (R = 1).$$

Now let us determine the affix τ_a of the centre I_a of the circle (I_a). The bisector of the interior angle B is BA_5 since point A_5 bisects arc CA. From this it follows that the bisector of the exterior angle B is the straight line $A_{12}B$ since the points A_5 and A_{12} are diametrically opposite points of the circle (O), and therefore $BA_5 \perp BA_{12}$. The bisector of the interior angle A is $AA_2 = A_7A_2$. To summarize: I_a is the point of intersection of the lines A_2A_7 and A_1A_{12}. Now set up the equations of these lines. The slope of A_2A_7 is 1, since $A_2A_7 \parallel A_1A_8$, and A_1A_8 is the real axis. Hence, the equation of the straight line A_2A_7 is of the form

$$z - \left(\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right) = \bar{z} - \left(\cos\frac{\pi}{7} - i\sin\frac{\pi}{7}\right)$$

or

$$z - \bar{z} = 2i\sin\frac{\pi}{7}. \tag{28}$$

The slope of the straight line A_1A_{12} is

$$\frac{a_{12} - 1}{\bar{a}_{12} - 1} = \frac{a_{12} - 1}{\dfrac{1}{a_{12}} - 1} = -a_{12}$$

and, consequently, the equation of A_1A_{12} is

$$z - 1 = -\left(\cos\frac{11\pi}{7} + i\sin\frac{11\pi}{7}\right)(\bar{z} - 1)$$

or

$$z - 1 = \left(\cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}\right)(\bar{z} - 1). \tag{29}$$

Solving the system (28)-(29), we find the affix τ_a of point I_a. From equation (28),

$$\bar{z} = z - 2i\sin\frac{\pi}{7}$$

and equation (29) takes the form

$$z - 1 = \left(\cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}\right)\left(z - 2i\sin\frac{\pi}{7} - 1\right),$$

$$z = 1 - \frac{2i\sin\frac{\pi}{7}\left(\cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}\right)}{2\sin^2\frac{2\pi}{7} - 2i\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}}$$

$$= 1 + \frac{i\sin\frac{\pi}{7}\left(\cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}\right)}{i\sin\frac{2\pi}{7}\left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)}$$

$$= 1 + \frac{\sin\frac{\pi}{7}\left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)}{\sin\frac{2\pi}{7}}$$

$$= 1 + \frac{\cos\frac{2\pi}{7}}{2\cos\frac{\pi}{7}} + i\sin\frac{\pi}{7} = \tau_a.$$

From this

$$OI_a^2 = |\tau_a|^2 = \frac{\left(2\cos\frac{\pi}{7} + \cos\frac{2\pi}{7}\right)^2}{4\cos^2\frac{\pi}{7}} + \sin^2\frac{\pi}{7}$$

$$= \frac{4\cos^2\frac{\pi}{7} + 4\cos\frac{\pi}{7}\cos\frac{2\pi}{7} + \cos^2\frac{2\pi}{7} + \sin^2\frac{2\pi}{7}}{4\cos^2\frac{\pi}{7}} =$$

$$=\frac{1+2\left(1+\cos\frac{2\pi}{7}\right)+2\left(\cos\frac{3\pi}{7}+\cos\frac{\pi}{7}\right)}{4\cos^2\frac{\pi}{7}}$$

$$=\frac{3+2\left(\cos\frac{2\pi}{7}-\cos\frac{4\pi}{7}-\cos\frac{6\pi}{7}\right)}{4\cos^2\frac{\pi}{7}}$$

$$=\frac{3+2\left(2\cos\frac{2\pi}{7}-\cos\frac{2\pi}{7}-\cos\frac{6\pi}{7}-\cos\frac{4\pi}{7}\right)}{4\cos^2\frac{\pi}{7}}.$$

But $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}=-\frac{1}{2}$, therefore

$$OI_a^2=\frac{3+4\cos\frac{2\pi}{7}+1}{4\cos^2\frac{\pi}{7}}=\frac{1+\cos\frac{2\pi}{7}}{\cos^2\frac{\pi}{7}}=2,$$

whence

$$OI_a=\sqrt{2}=R\sqrt{2}=OH.$$

2°. $r_a=p\tan\frac{A}{2}=p\tan\frac{\pi}{14}\left(\sin\frac{\pi}{7}+\sin\frac{2\pi}{7}+\sin\frac{3\pi}{7}\right)^*.$

Let

$$y=\sin\frac{\pi}{7}+\sin\frac{2\pi}{7}+\sin\frac{3\pi}{7}.$$

Multiplying both sides of this equation by $2\sin\left(\frac{\pi}{14}\right)$, we obtain

$$2y\sin\frac{\pi}{14}=2\sin\frac{\pi}{14}\sin\frac{\pi}{7}+2\sin\frac{\pi}{14}\sin\frac{2\pi}{7}+2\sin\frac{\pi}{14}\sin\frac{3\pi}{7}$$

$$=\cos\frac{\pi}{14}-\cos\frac{3\pi}{14}+\cos\frac{3\pi}{14}-\cos\frac{5\pi}{14}+\cos\frac{5\pi}{14}-\cos\frac{7\pi}{14}=\cos\frac{\pi}{14}.$$

* If the radius of the circle is equal to 1, the chord subtending arc α is equal to $2\sin(\alpha/2)$. Angle A is an inscribed angle intercepting the arc $2\pi/7$, hence $A_1A_3=BC=$ $=2\sin(\pi/7)$ and similarly for the other two sides. The lengths of the sides may also be found by the sine theorem: $a=2R\sin A=2\sin(\pi/7)$ and so forth.

Hence,

$$y = \frac{1}{2} \cot \frac{\pi}{14}$$

and so

$$r_a = \tan \frac{\pi}{14} \cdot \frac{1}{2} \cot \frac{\pi}{14} = \frac{1}{2} = \frac{R}{2},$$

whence

$$R = 2r_a.$$

3°. Furthermore, $I_aM = h - \tau_a$ (h and τ_a were computed in item 1°), whence

$$h - \tau_a = \cos \frac{2\pi}{7} + \cos \frac{6\pi}{7} - \frac{\cos \frac{2\pi}{7}}{2 \cos \frac{\pi}{7}} + i \sin \frac{2\pi}{7}$$

$$= \frac{2 \cos \frac{\pi}{7} \cos \frac{2\pi}{7} + 2 \cos \frac{6\pi}{7} \cos \frac{\pi}{7} - \cos \frac{2\pi}{7}}{2 \cos \frac{\pi}{7}} + i \sin \frac{2\pi}{7}$$

$$= \frac{\cos \frac{\pi}{7} + \cos \frac{3\pi}{7} + \cos \frac{5\pi}{7} - 1 - \cos \frac{2\pi}{7}}{2 \cos \frac{\pi}{7}} + i \sin \frac{2\pi}{7}$$

$$= \frac{-\cos \frac{6\pi}{7} - \cos \frac{4\pi}{7} - \cos \frac{2\pi}{7} - 1 - \cos \frac{2\pi}{7}}{2 \cos \frac{\pi}{7}} + i \sin \frac{2\pi}{7}$$

$$= \frac{\frac{1}{2} - 1 - \cos \frac{2\pi}{7}}{2 \cos \frac{\pi}{7}} + i \sin \frac{2\pi}{7} = \frac{-\frac{1}{2} - \cos \frac{2\pi}{7}}{2 \cos \frac{\pi}{7}} + i \sin \frac{2\pi}{7}$$

$$= \frac{\cos \frac{4\pi}{7} + \cos \frac{6\pi}{7}}{2 \cos \frac{\pi}{7}} + i \sin \frac{2\pi}{7} = \frac{2 \cos \frac{5\pi}{7} \cos \frac{\pi}{7}}{2 \cos \frac{\pi}{7}} + i \sin \frac{2\pi}{7}$$

$$= \cos \frac{5\pi}{7} + i \sin \frac{2\pi}{7} = \cos \frac{5\pi}{7} + i \sin \frac{5\pi}{7} = a_6.$$

Thus,

$$h - \tau_a = \cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7} = a_6$$

and so

$$HI_a = |h - \tau_a| = 1 = R.$$

4°. $a^2 + b^2 + c^2 = 4\sin^2\frac{\pi}{7} + 4\sin^2\frac{2\pi}{7} + 4\sin^2\frac{3\pi}{7}$

$$= 2\left(1 - \cos\frac{2\pi}{7} + 1 - \cos\frac{4\pi}{7} + 1 - \cos\frac{6\pi}{7}\right)$$

$$= 2\left[3 - \left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right)\right] = 2[3 - (-1/2)] = 7 = 7R^2.$$

5°. The directed line segments $\overrightarrow{I_aH}$ and $\overrightarrow{OA_6}$ are equivalent, that is to say, they are parallel, equal in magnitude, and in the same direction; since $h - \tau_a = a_6$ (see item 3°); hence I_aHA_6O is a parallelogram. The centre of this parallelogram is the midpoint of the line segment OH, that is, the centre of the Euler circle of triangle ABC.

6°. The affix of the point P is

$$\frac{h + a_6}{2} = \frac{1}{2}\left[1 + \cos\frac{2\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{5\pi}{7}\right.$$

$$\left. + i\left(\sin\frac{2\pi}{7} + \sin\frac{6\pi}{7} + \sin\frac{5\pi}{7}\right)\right]$$

$$= \frac{1}{2}\left[1 - \cos\frac{\pi}{7} + i\left(2\sin\frac{2\pi}{7} + \sin\frac{\pi}{7}\right)\right].$$

whence

$$\left|\frac{h + a_6}{2}\right|^2 = \frac{1}{4}\left[\left(1 - \cos\frac{\pi}{7}\right)^2 + \left(2\sin\frac{2\pi}{7} + \sin\frac{\pi}{7}\right)^2\right]$$

$$= \frac{1}{4}\left(1 - 2\cos\frac{\pi}{7} + \cos^2\frac{\pi}{7} + 4\sin^2\frac{2\pi}{7} + 4\sin\frac{2\pi}{7}\sin\frac{\pi}{7} + \sin^2\frac{\pi}{7}\right)$$

$$= \frac{1}{4}\left[2 - 2\cos\frac{\pi}{7} + 2\left(1 - \cos\frac{4\pi}{7}\right) + 2\left(\cos\frac{\pi}{7} - \cos\frac{3\pi}{7}\right)\right]$$

$$= \frac{1}{2}\left(1 - \cos\frac{\pi}{7} + 1 - \cos\frac{4\pi}{7} + \cos\frac{\pi}{7} - \cos\frac{3\pi}{7}\right) = \frac{1}{2}\cdot 2 = 1,$$

$$\left|\frac{h + a_6}{2}\right| = 1,$$

That is, the midpoint P of segment HA_6 lies on (O). The centre O_9 of the Euler circle of $\triangle ABC$ is the midpoint of segment OH. Thus, O_9P is the midline of triangle OHA_6 and, hence, $O_9P = \frac{1}{2} OA_6 = \frac{R}{2}$. From this it follows that point P also lies on the Euler circle (O_9).

7°. We will prove that $\triangle \overrightarrow{AI_aH}$ and $\triangle \overrightarrow{HBI_a}$ are similar and have the same orientation. We have

$$\Delta = \begin{vmatrix} a_7 & h & 1 \\ \tau_a & 1 & 1 \\ h & \tau_a & 1 \end{vmatrix} = \begin{vmatrix} a_7 - h & h - \tau_a & 0 \\ \tau_a - h & 1 - \tau_a & 0 \\ h & \tau_a & 1 \end{vmatrix} = (a_7 - h)(1 - \tau_a) + (h - \tau_a)^2.$$

Furthermore,

$$a_7 - h = \cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7} - 1 - \cos\frac{2\pi}{7} - \cos\frac{6\pi}{7} - i\sin\frac{2\pi}{7} - i\sin\frac{6\pi}{7}$$

$$= -1 - \cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7},$$

$$1 - \tau_a = 1 - 1 - \frac{\cos\frac{2\pi}{7}}{2\cos\frac{\pi}{7}} - i\sin\frac{\pi}{7} = -\frac{\cos\frac{2\pi}{7}}{2\cos\frac{\pi}{7}} - i\sin\frac{\pi}{7},$$

$$(h - \tau_a)^2 = a_6^2 = \cos\frac{10\pi}{7} + i\sin\frac{10\pi}{7} = -\cos\frac{3\pi}{7} - i\sin\frac{3\pi}{7},$$

$$(a_7 - h)(1 - \tau_a) = \left(1 + \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)\left(\frac{\cos\frac{2\pi}{7}}{2\cos\frac{\pi}{7}} + i\sin\frac{\pi}{7}\right)$$

$$= \left(1 + \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)\frac{\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}}{2\cos\frac{\pi}{7}}$$

$$= \left(2\cos^2\frac{\pi}{7} + 2i\sin\frac{\pi}{7}\cos\frac{\pi}{7}\right)\frac{\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}}{2\cos\frac{\pi}{7}} = \cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}.$$

Consequently,

$$\Delta = (h - \tau_a)^2 + (a_7 - h)(1 - \tau_a) = 0.$$

To summarize: $\overrightarrow{AI_aH} \downarrow\downarrow \overrightarrow{HBI_a}$ and the triangles $\overrightarrow{AI_aH}$ and $\overrightarrow{HBI_a}$ are similar and have the same orientation (precisely for the order in which the vertices are specified).

We can similarly prove that $\triangle\ \overrightarrow{AI_aH}$ and $\triangle\ \overrightarrow{I_aHC}$ are similar and have the same orientation.

8°. We will prove, for example, that point B^*, which is symmetric to point $B = A_1$ with respect to the bisector of the interior angle A, lies on the straight line HI_a. The equation of the bisector of the interior angle A is

$$z - \overline{z} = 2i \sin \frac{\pi}{7}\cdot$$

The equation of the perpendicular dropped from point $A_1 = B$ on that bisector is of the form

$$z + \overline{z} = 2,$$

whence we find the affix b_1 of the projection B_1 of point B on the bisector of angle A:

$$b_1 = 1 + i \sin \frac{\pi}{7}\cdot$$

The affix b* of point B^* can be found from the relation

$$\frac{b^* + 1}{2} = b_1,$$

whence

$$b^* = 1 + 2i \sin \frac{\pi}{7}\cdot$$

Furthermore,

$$h - \tau_a = \cos \frac{5\pi}{7} + i \sin \frac{5\pi}{7},$$

$$\tau_a - b^* = 1 + \frac{\cos \frac{2\pi}{7}}{2 \cos \frac{\pi}{7}} + i \sin \frac{\pi}{7} - 1 - 2i \sin \frac{\pi}{7}$$

$$= \frac{\cos \frac{2\pi}{7}}{2 \cos \frac{\pi}{7}} - i \sin \frac{\pi}{7} = \frac{\cos \frac{2\pi}{7} - i \sin \frac{2\pi}{7}}{2 \cos \frac{\pi}{7}} = -\frac{\cos \frac{5\pi}{7} + i \sin \frac{5\pi}{7}}{2 \cos \frac{\pi}{7}}$$

and, hence, the ratio

$$\frac{h-\tau_a}{\tau_a-b^*} = -2\cos\frac{\pi}{7}$$

is a real number, and so the points H, I_a, B^* lie on one straight line. It is left to the reader to prove the other two propositions of this item.

9°. Let us consider the triangle OI_aA_6, We have

$$OA_6^2 = 1,$$

$$OI_a^2 = |\tau_a|^2 = \left(1+\frac{\cos\frac{2\pi}{7}}{2\cos\frac{\pi}{7}}\right)^2 + \sin^2\frac{\pi}{7}$$

$$= 1+\frac{\cos\frac{2\pi}{7}}{\cos\frac{\pi}{7}}+\frac{\cos^2\frac{2\pi}{7}}{4\cos^2\frac{\pi}{7}}+\sin^2\frac{\pi}{7}$$

$$= \frac{4\cos^2\frac{\pi}{7}+4\cos\frac{\pi}{7}\cos\frac{2\pi}{7}+\cos^2\frac{2\pi}{7}+4\cos^2\frac{\pi}{7}\sin^2\frac{\pi}{7}}{4\cos^2\frac{\pi}{7}}$$

$$= \frac{1+2\left(1+\cos\frac{2\pi}{7}\right)+2\left(\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}\right)}{4\cos^2\frac{\pi}{7}}$$

$$= \frac{3+2\left(\cos\frac{2\pi}{7}-\cos\frac{4\pi}{7}-\cos\frac{6\pi}{7}\right)}{4\cos^2\frac{\pi}{7}}$$

$$= \frac{3+2\left(\cos\frac{2\pi}{7}+\cos\frac{2\pi}{7}+\frac{1}{2}\right)}{4\cos^2\frac{\pi}{7}} = \frac{4+4\cos\frac{2\pi}{7}}{4\cos^2\frac{\pi}{7}} = 2.$$

And, finally, from the parallelogram OI_aHA_6 we have (the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the four sides):

$$OI_a^2 + HA_6^2 + OA_6^2 + I_aH^2 = OH^2 + I_aA_6^2.$$

But $OI_a^2 = HA_6^2 = 2$ (see items 1° and 5°), $OA_6 = I_aH = 1$, $OH = \sqrt{2}$; consequently,

$$2 + 2 + 1 + 1 = 2 + I_aA_6^2,$$

whence

$$I_a A_6^2 = 4.$$

The triangles OI_aA_6 and A_6I_aH are equal, and so the squares of the their sides I_aH^2, HA_6^2, $I_aA_6^2$, also form a geometric progression with ratio 2.

Problem 12. Given the linear fractional function

$$u = \frac{az+b}{cz+d}, \tag{30}$$

where a, b, c, d are fixed complex numbers and $ad - bc \neq 0$, $c \neq 0$; z is the argument and u the value of the function.

Prove that if $|c| \neq |d|$ then the image of the unit circle under this transformation is a circle; find its radius and the affix of the centre.

Solution. Transform the function u as follows:

$$u = \frac{a}{c} + \frac{az+b}{cz+d} - \frac{a}{c} = \frac{a}{c} + \frac{bc-ad}{c(cz+d)} = \frac{a}{c} + \frac{\dfrac{bc-ad}{c^2}}{z+\dfrac{d}{c}}.$$

(1) The transformation $z \to z + \dfrac{d}{c}$ takes the unit circle $(\Omega_1) = (0, 1)$* into the circle $(\Omega_2) = \left(\dfrac{d}{c}, 1\right)$ (Fig. 15).

(2) The transformation $z + \dfrac{d}{c} \to \dfrac{1}{z+\dfrac{d}{c}}$ consists in symmetry with respect to the real axis Ox, under which symmetry the circle $(\Omega_2)=\left(\dfrac{d}{c}, 1\right)$ goes into the circle $(\Omega_3) = \left(\dfrac{\bar{d}}{\bar{c}}, 1\right)$, and subsequent inversion (see chapter IV) of (Ω_3) with respect to (Ω_1), under which inversion the circle (Ω_3) goes into the circle (Ω_4); to construct (Ω_4) it suffices to draw the straight line $O\Omega_3$ and to construct the images P' and Q' of the endpoints P and Q of the diameter that cuts out that line on (Ω_3). The circle constructed on segment $P'Q'$ as a diameter is precisely the circle (Ω_4). Let us compute

* The symbol (z_0, R) will be used to denote a circle of radius R, the affix of whose centre is z_0.

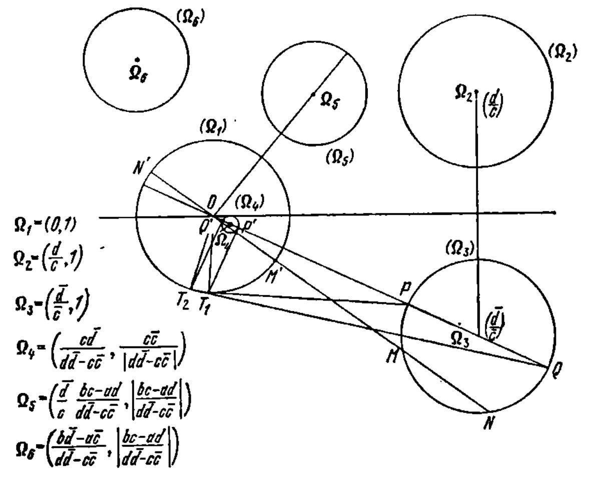

Fig. 15

the radius of (Ω_4) and the affix of its centre. To do this, note that (Ω_3) passes into (Ω_4) under the inversion with respect to the circle (Ω_1) and under the homothetic transformation $(0,\ 1/\sigma)$, where σ is the power of the point O with respect to the circle (Ω_3).

But under the homothetic transformation $(O,\ 1/\sigma)$, the centre of (Ω_3) goes into the centre of (Ω_4); hence the affix ω_4 of the centre Ω_4 of (Ω_4) is equal to

$$\omega_4 = \frac{1}{\sigma}\frac{\bar d}{\bar c}$$

and the radius is equal to the product of the radius $R_3 = 1$ of the circle (Ω_3) by the modulus of the homothetic ratio, that is, by $1/|\sigma|$:

$$R_4 = R_3\frac{1}{|\sigma|} = \frac{1}{|\sigma|} \quad (R_3 = 1).$$

Furthermore,

$$\sigma = O\Omega_3^2 - R_3^2 = \frac{\bar d}{\bar c}\frac{d}{c} - 1 = \frac{d\bar d - c\bar c}{c\bar c}.*$$

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* In the general case, under inversion with respect to the point O, we have

$$(OM)\cdot(OM') = k,$$

where $k \neq 0$ is the power of the inversion, and, hence,

$$\frac{(OM')}{(ON)} = \frac{k}{\sigma},$$

where k is the second point of intersection of the straight line OM with circle being inverted; $\sigma \neq 0$, since $|d| \neq |c|$.

so that

$$(\Omega_4) = \left(\frac{c\bar{d}}{d\bar{d} - c\bar{c}},\ \frac{c\bar{c}}{|d\bar{d} - c\bar{c}|}\right).$$

(3) The transformation

$$\frac{1}{z + \dfrac{d}{c}} \to, \frac{\dfrac{bc - ad}{c^2}}{z + \dfrac{d}{c}}$$

consists in turning the plane through an angle of arg $\dfrac{bc - ad}{c^2}$ and in a homothetic transformation with centre O and ratio $\dfrac{|bc - ad|}{|c|^2}$. After these two transformations, the circle (Ω_4) goes into the circle (Ω_5), the affix ω_5 of whose centre is

$$\omega_5 = \omega_4 \frac{bc - ad}{c^2} = \frac{c\bar{d}}{d\bar{d} - c\bar{c}} \frac{bc - ad}{c^2} = \frac{\bar{d}}{c} \frac{bc - ad}{d\bar{d} - c\bar{c}}$$

and the radius is

$$R_5 = \left| \frac{c\bar{c}}{d\bar{d} - c\bar{c}} \cdot \frac{bc - ad}{c^2} \right| = \left| \frac{bc - ad}{d\bar{d} - c\bar{c}} \right|^* .$$

Thus

$$(\Omega_5) = \left(\frac{\bar{d}}{c} \frac{bc - ad}{d\bar{d} - c\bar{c}},\ \left|\frac{bc - ad}{d\bar{d} - c\bar{c}}\right| \right).$$

(4) Finally, the transformation of translation

$$\frac{\dfrac{bc - ad}{c^2}}{z + \dfrac{d}{c}} \to \frac{a}{c} + \frac{\dfrac{bc - ad}{c^2}}{z + \dfrac{d}{c}}$$

carries the circle (Ω_5) into (Ω_6) of the same radius: $R_6 = R_5$, and the affix ω_6 of the centre Ω_6 of (Ω_6) is equal to

$$\omega_6 = \frac{a}{c} + \frac{\bar{d}}{c} \frac{bc - ad}{d\bar{d} - a\bar{c}} = \frac{b\bar{d} - a\bar{c}}{d\bar{d} - c\bar{c}} .$$

---

* $\left|\dfrac{c\bar{c}}{c^2}\right| = \left|\dfrac{|c|^2}{c^2}\right| = \dfrac{|c|^2}{|c^2|} = \dfrac{|c|^2}{|c|^2} = 1.$

To summarize, then: as a result of the transformation

$$u = \frac{az+b}{cz+d}, \quad ad-bc \neq 0, \quad c \neq 0, \quad |c| \neq |d|,$$

the circle $(\Omega_1) = (0,\ 1)$ goes into the circle

$$(\Omega_6) = \left(\frac{b\bar{d}-a\bar{c}}{d\bar{d}-c\bar{c}},\quad \left|\frac{bc-ad}{d\bar{d}-c\bar{c}}\right|\right).$$

Problem 13. Suppose $\overrightarrow{BCA_1A_2}$, $\overrightarrow{CAB_1B_2}$, $\overrightarrow{ABC_1C_2}$ are squares with the same orientation and constructed on the sides BC, CA, AB of $\triangle\ ABC$, the orthocentre of which is H. Denote by $(O) = (ABC)$ the circle passing through the points A, B, C. Let P, Q, R be the respective centres of the squares $\overrightarrow{A_1B_2C'C''}$, $\overrightarrow{B_1C_2A'A''}$, $\overrightarrow{C_1A_2B'B''}$ having the same orientation as the first three squares. Denote by P_1, P_2, P_3 the orthogonal projections of point P on the sides BC, CA, AB of $\triangle ABC$, by Q_1, Q_2, Q_3 the orthogonal projections of point Q on the same sides and by R_1, R_2, R_3 the orthogonal projections of point R on the same sides BC, CA, AB of $\triangle ABC$. Prove that the sum of the *forces*

$$\overrightarrow{PP_1} + \overrightarrow{PP_2} + \overrightarrow{PP_3} + \overrightarrow{QQ_1} + \overrightarrow{QQ_2} + \overrightarrow{QQ_3} + \overrightarrow{RR_1} + \overrightarrow{RR_2} + \overrightarrow{RR_3} \quad (31)$$

is equal to the vector $\overrightarrow{OH'}$, where H' is the orthocentre of the triangle whose vertices are the feet of the altitudes of the given triangle ABC.

Prove that the straight line on which the resultant of these forces * lies passes through the endpoint of the directed line segment $4\overrightarrow{OH}$.

Now if the squares of the second triplet have an orientation opposite that of the first three squares, then the sum of the forces (31) belongs to the straight line OH'.

Set up the equations of these two straight lines and take the circle $(O) = (ABC)$ as the unit circle (Fig. 16).

Solution. 1°. Let a, b, c be the affixes of the points A, B, C. Find the affixes p, q, r of the points P, Q, R. Denoting by a_1, b_1, c_1, a_2, b_2, c_2 the affixes of the points A_1, B_1, C_1, A_2, B_2, C_2, we have

$$\left.\begin{aligned} b_2 &= c + \overrightarrow{CB_2} = c + i\overrightarrow{CA} = c + i(a-c) = ia + (1-i)\,c, \\ a_1 &= c + \overrightarrow{CA_1} = c - i\overrightarrow{CB} = c - i\,(b-c) = (1+i)\,c - ib. \end{aligned}\right\} \quad (32)$$

Also,

$$p = b_2 + \frac{1}{2}\overrightarrow{B_2A_1} - \frac{i}{2}\overrightarrow{B_2A_1} = b_2 + \frac{1}{2}(a_1-b_2) - \frac{i}{2}(a_1-b_2)$$

$$= b_2 + \frac{1}{2}a_1 - \frac{1}{2}b_2 - \frac{ia_1}{2} + \frac{ib_2}{2} = \frac{1+i}{2}b_2 + \frac{1-i}{2}a_1$$

* Force is a sliding vector.

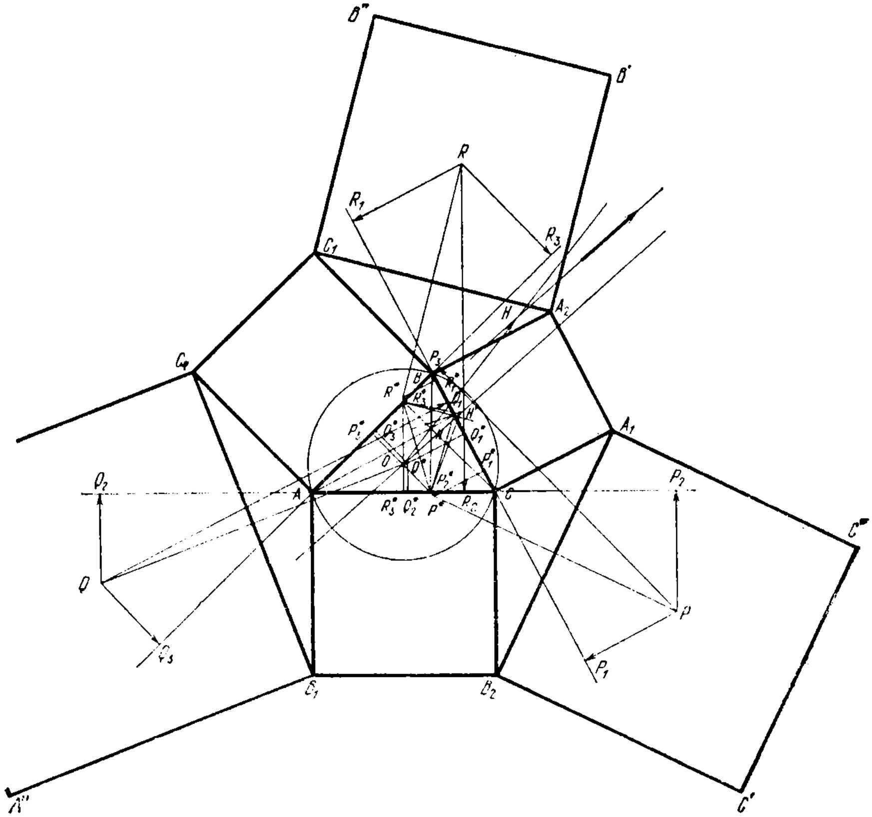

Fig. 16

and, taking into account (32), we have

$$p=\frac{1+i}{2}[ia+(1-i)c]+\frac{1-i}{2}[(1+i)c-ib)]=2c+\frac{i-1}{2}a-\frac{1+i}{2}b. \tag{33}$$

Similarly (we can carry out a circular permutation of the letters a, b, c), we can find

$$\left.\begin{aligned} q&=2a+\frac{i-1}{2}b-\frac{1+i}{2}c,\\ r&=2b+\frac{i-1}{2}c-\frac{1+i}{2}a. \end{aligned}\right\} \tag{34}$$

Note here that it follows from (33) and (34) that

$$p+q+r=a+b+c=\sigma_1$$

and, hence, $\triangle ABC$ and $\triangle PQR$ have a common centroid (the point of intersection of the medians).

Now, from the equations of the straight lines BC and PP_1,

$$z - b = -bc(\bar{z} - \bar{b}),$$
$$z - p = bc(\bar{z} - \bar{p})$$

or

$$z + bc\bar{z} = b + c,$$
$$z - bc\bar{z} = p - bc\bar{p},$$

we find the affix p_1 of point P_1:

$$p_1 = \frac{1}{2}(b + c + p - bc\bar{p});$$

and, hence,

$$\overrightarrow{PP_1} = p_1 - p = \frac{1}{2}(b + c + p - bc\bar{p}) - p = \frac{1}{2}(b + c - p - bc\bar{p}).$$

Similarly,

$$\overrightarrow{PP_2} = p_2 - p = \frac{1}{2}(c + a - p - ca\bar{p}),$$
$$\overrightarrow{PP_3} = p_3 - p = \frac{1}{2}(a + b - p - ab\bar{p}),$$

whence

$$\overrightarrow{PP_1} + \overrightarrow{PP_2} + \overrightarrow{PP_3} = \sigma_1 - \frac{3}{2}p - \frac{1}{2}\sigma_2\bar{p}.$$

Similarly,

$$\overrightarrow{QQ_1} + \overrightarrow{QQ_2} + \overrightarrow{QQ_3} = \sigma_1 - \frac{3}{2}q - \frac{1}{2}\sigma_2\bar{q},$$
$$\overrightarrow{RR_1} + \overrightarrow{RR_2} + \overrightarrow{RR_3} = \sigma_1 - \frac{3}{2}r - \frac{1}{2}\sigma_2\bar{r}.$$

Adding these relations term by term, we can find the principal vector of the resultant:

$$h' = \overrightarrow{PP_1} + \overrightarrow{PP_2} + \overrightarrow{PP_3} + \overrightarrow{QQ_1} + \overrightarrow{QQ_2} + \overrightarrow{QQ_3} + \overrightarrow{RR_1} + \overrightarrow{RR_2} + \overrightarrow{RR_3}$$

$$3\sigma_1 - \frac{3}{2}(p + q + r) - \frac{1}{2}\sigma_2(\bar{p} + \bar{q} + \bar{r}) = 3\sigma_1 - \frac{9}{2}\frac{p + q + r}{3} -$$

$$-\frac{3}{2}\sigma_2\frac{\bar{p}+\bar{q}+\bar{r}}{3}=3\sigma_1-\frac{9}{2}\frac{a+b+c}{3}-\frac{3}{2}\sigma_2\frac{\bar{a}+\bar{b}+\bar{c}}{3}$$

$$=3\sigma_1-\frac{9}{2}g-\frac{3}{2}\sigma_2\bar{g}=3\left(\sigma_1-\frac{3}{2}g-\frac{1}{2}\sigma_2\bar{g}\right),$$

where $g=(a+b+c)/3$ is the affix of the point of intersection of the medians of $\triangle ABC$ (or, what is the same, the point of intersection of the medians of $\triangle PQR$).

But since $a+b+c=\sigma_1$, the sum h' may be transformed as follows:

$$h'=3\left(\sigma_1-\frac{3}{2}\frac{\sigma_1}{3}-\frac{1}{2}\sigma_2\frac{\bar{\sigma}_1}{3}\right)=3\left(\frac{\sigma_1}{2}-\frac{\sigma_2\bar{\sigma}_1}{6}\right)=\frac{1}{2}(3\sigma_1-\sigma_2\bar{\sigma}_1).$$

Let us now prove that h' is the affix of the orthocentre of $\triangle A_hB_hC_h$ formed by the feet of the altitudes of the given triangle. The equations of BC and of the altitude from A to BC are of the form

$$z+bc\bar{z}=b+c,$$

$$z-bc\bar{z}=a-\frac{bc}{a}.$$

Adding, we find the affix a_h of point A_h:

$$a_h=\frac{1}{2}\left(\sigma_1-\frac{bc}{a}\right)$$

and, similarly,

$$b_h=\frac{1}{2}\left(\sigma_1-\frac{ca}{b}\right),\quad c_h=\frac{1}{2}\left(\sigma_1-\frac{ab}{c}\right).$$

The slope of B_hC_h is

$$\frac{b_h-c_h}{\bar{b}_h-\bar{c}_h}=\frac{\dfrac{ab}{c}-\dfrac{ac}{b}}{\dfrac{c}{ab}-\dfrac{b}{ac}}=\frac{\dfrac{a(b^2-c^2)}{bc}}{\dfrac{c^2-b^2}{abc}}=-a^2.$$

The equation of the altitude dropped from vertex A_h to side B_hC_h is

$$z-\frac{1}{2}\left(\sigma_1-\frac{bc}{a}\right)=a^2\left[\bar{z}-\frac{1}{2}\left(\bar{\sigma}_1-\frac{a}{bc}\right)\right]$$

or

$$z-a^2\bar{z}=\frac{1}{2}\sigma_1-\frac{bc}{2a}-\frac{1}{2}a^2\bar{\sigma}_1+\frac{a^3}{2bc}$$

or

$$z - a^2\bar{z} = \frac{1}{2}\sigma_1 - \frac{1}{2}a^2\bar{\sigma}_1 + \frac{a^4 - b^2c^2}{2\sigma_3}. \tag{35}$$

In similar fashion we can write down the equation of the altitude from B_h to C_hA_h:

$$z - b^2\bar{z} = \frac{1}{2}\sigma_1 - \frac{1}{2}b^2\bar{\sigma}_1 + \frac{b^4 - c^2a^2}{2\sigma_3}. \tag{36}$$

Subtracting (36) from (35) term by term, we obtain

$$(b^2 - a^2)\,\bar{z} = \frac{1}{2}(b^2 - a^2)\,\bar{\sigma}_1 - \frac{b^4 - a^4 + c^2(b^2 - a^2)}{2\sigma_3}$$

or, cancelling $b^2 - a^2$,

$$\bar{z} = \bar{h}' = \frac{1}{2}\bar{\sigma}_1 - \frac{a^2 + b^2 + c^2}{2\sigma_3} = \frac{1}{2}\bar{\sigma}_1 - \frac{\sigma_1^2 - 2\sigma_2}{2\sigma_3}$$

$$= \frac{1}{2}\bar{\sigma}_1 - \frac{\sigma_1^2}{2\sigma_3} + \frac{\sigma_2}{\sigma_3} = \frac{1}{2}\bar{\sigma}_1 - \frac{\bar{\sigma}_2\sigma_1}{2} + \bar{\sigma}_1$$

$$= \frac{3}{2}\bar{\sigma}_1 - \frac{\bar{\sigma}_2\sigma_1}{2} = \frac{1}{2}(3\bar{\sigma}_1 - \sigma_1\bar{\sigma}_2)$$

and, hence,

$$h' = \frac{1}{2}(3\sigma_1 - \bar{\sigma}_1\sigma_2).$$

We now set up the equation of the straight line to which the resultant thus found belongs. Since the nine indicated forces emanate three at a time from a single point, their resultant is equal to the sum of three forces:

$$\overrightarrow{PP'} = \overrightarrow{PP_1} + \overrightarrow{PP_2} + \overrightarrow{PP_3},$$

$$\overrightarrow{QQ'} = \overrightarrow{QQ_1} + \overrightarrow{QQ_2} + \overrightarrow{QQ_3},$$

$$\overrightarrow{RR'} = \overrightarrow{RR_1} + \overrightarrow{RR_2} + \overrightarrow{RR_3},$$

these forces are laid off, respectively, from the points P, Q, R. From the relation

$$p' - p = \sigma_1 - \frac{3}{2}p - \frac{1}{2}\sigma_2\bar{p}$$

we find the affix p' of point P' and, similarly, the affixes of the points Q' and R':

$$p' = \sigma_1 - \frac{1}{2}p - \frac{1}{2}\sigma_2\bar{p},$$

$$q' = \sigma_1 - \frac{1}{2}q - \frac{1}{2}\sigma_2\bar{q},$$

$$r' = \sigma_1 - \frac{1}{2}r - \frac{1}{2}\sigma_2\bar{r}.$$

Remark. If on a plane (which we consider oriented by the introduction of a rectangular coordinate system) there is given a set of forces $\overrightarrow{A_kB_k}$ $(k = 1, 2, \ldots, n)$, then the principal vector of their resultant can be found as the sum of the free vectors:

$$\mathbf{K} = \sum_{k=1}^{n}\overrightarrow{A_kB_k}.$$

The straight line to which the sum of forces belongs is a locus of points $M(x, y)$ for which

$$\sum_{k=1}^{n}\mathrm{mom}_M\,\overrightarrow{A_kB_k} = 0.$$

Since, in the plane, it is natural to regard as the moment of force $\mathbf{F}$ the cross product

$$\mathrm{mom}_M\mathbf{F} = (\overrightarrow{MT}, \mathbf{F}),$$

where T is the point of application of the force $\mathbf{F}$, it follows that the equation of the straight line to which the resultant of the system of forces $\overrightarrow{A_kB_k}$ belongs is of the form

$$\sum_{k=1}^{n}\begin{vmatrix} x & y & 1 \\ a_k & a'_k & 1 \\ b_k & b'_k & 1 \end{vmatrix} = 0, \tag{37}$$

where $(a_k, a'_k) = A_k$, $(b_k, b'_k) = B_k$.

Equation (37) is, generally, an equation of the first degree; hence, (37) is, generally speaking, an equation of a straight line [the left-hand member of (37) vanishes if and only if the principal vector of the system of forces is zero].

The equation of the straight line containing the vector of the sum of forces $\overrightarrow{A_kB_k}$ may be written differently by introducting the *affixes* a_k and b_k of points A_k and B_k:

$$\sum_{k=1}^{n} \begin{vmatrix} z & \bar{z} & 1 \\ a_k & \bar{a}_k & 1 \\ b_k & \bar{b}_k & 1 \end{vmatrix} = 0. \tag{38}$$

Applying this equation to the given problem, we have

$$\begin{vmatrix} z & \bar{z} & 1 \\ p & \bar{p} & 1 \\ p' & \bar{p}' & 1 \end{vmatrix} = \begin{vmatrix} z & p & \sigma_1 - \frac{1}{2}p - \frac{1}{2}\sigma_2\bar{p} \\ \bar{z} & \bar{p} & \bar{\sigma}_1 - \frac{1}{2}\bar{p} - \frac{1}{2}\bar{\sigma}_2 p \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} z & p & \sigma_1 - \frac{1}{2}\sigma_2\bar{p} \\ \bar{z} & \bar{p} & \bar{\sigma}_1 - \frac{1}{2}\bar{\sigma}_2 p \\ 1 & 1 & \frac{3}{2} \end{vmatrix}$$

$$= z\left(\frac{3}{2}\bar{p} - \bar{\sigma}_1 + \frac{1}{2}\bar{\sigma}_2 p\right) + \bar{z}\left(-\frac{3}{2}p + \sigma_1 - \frac{1}{2}\sigma_2\bar{p}\right)$$
$$+ p\bar{\sigma}_1 - \bar{p}\sigma_1 + \frac{1}{2}\sigma_2\bar{p}^2 - \frac{1}{2}\bar{\sigma}_2 p^2.$$

Adding together three similar expressions that result from this one by substituting q for p and then r for q, and equating this sum to zero, we obtain the equation of the desired line in the form (also note that $d + q + r = a + b + c = \sigma_1$)

$$z\left(\frac{3}{2}\bar{\sigma}_1 - 3\bar{\sigma}_1 + \frac{1}{2}\bar{\sigma}_2\sigma_1\right) + \bar{z}\left(-\frac{3}{2}\sigma_1 + 3\sigma_1 - \frac{1}{2}\sigma_2\bar{\sigma}_1\right)$$
$$+ \frac{1}{2}\sigma_2(\bar{p}^2 + \bar{q}^2 + \bar{r}^2) - \frac{1}{2}\bar{\sigma}_2(p^2 + q^2 + r^2) = 0$$

or

$$\left(-\frac{3}{2}\bar{\sigma}_1 + \frac{1}{2}\sigma_1\bar{\sigma}_2\right)z + \left(\frac{3}{2}\sigma_1 - \frac{1}{2}\bar{\sigma}_1\sigma_2\right)\bar{z}$$
$$+ \frac{1}{2}\sigma_2(\bar{p}^2 + \bar{q}^2 + \bar{r}^2) - \frac{1}{2}\bar{\sigma}_2(p^2 + q^2 + r^2) = 0$$

or

$$(3\bar{\sigma}_1 - \sigma_1\bar{\sigma}_2)z - (3\sigma_1 - \bar{\sigma}_1\sigma_2)\bar{z} + \bar{\sigma}_2(p^2 + q^2 + r^2) - \sigma_2(\bar{p}^2 + \bar{q}^2 + \bar{r}^2) = 0.$$

We then find

$$p^2 = 4c^2 - \frac{ia^2}{2} + \frac{ib^2}{2} + 2ac(i-1) - 2cb(i+1) + ab,$$

$$q^2 = 4a^2 - \frac{ib^2}{2} + \frac{ic^2}{2} + 2ba(i-1) - 2ac(i+1) + bc,$$

$$r^2 = 4b^2 - \frac{ic^2}{2} + \frac{ia^2}{2} + 2cb(i-1) - 2ba(i+1) + ca,$$

and, hence,

$$p^2 + q^2 + r^2 = 4(\sigma_1^2 - 2\sigma_2) - 3\sigma_2 = 4\sigma_1^2 - 11\sigma_2.$$

From this it follows that

$$\bar{p}^2 + \bar{q}^2 + \bar{r}^2 = 4\bar{\sigma}_1^2 - 11\bar{\sigma}_2$$

and the equation of the straight line assumes the form

$$(3\bar{\sigma}_1 - \sigma_1\bar{\sigma}_2)\, z - (3\sigma_1 - \bar{\sigma}_1\sigma_2)\bar{z} + \bar{\sigma}_2(4\sigma_1^2 - 11\sigma_2) - \sigma_2(4\bar{\sigma}_1^2 - 11\bar{\sigma}_2) = 0$$

or

$$(3\bar{\sigma}_1 - \sigma_1\bar{\sigma}_2)\, z - (3\sigma_1 - \sigma_2\bar{\sigma}_1)\bar{z} + 4(\sigma_1^2\bar{\sigma}_2 - \sigma_2\bar{\sigma}_1^2) = 0. \tag{39}$$

If instead of z we put into the left-hand member the affix of the endpoint of the directed line segment $\overrightarrow{4OH}$, that is, $4\sigma_1$, we obtain

$$4(3\bar{\sigma}_1 - \sigma_1\bar{\sigma}_2)\, \sigma_1 - 4(3\sigma_1 - \sigma_2\bar{\sigma}_1)\, \bar{\sigma}_1 + 4\sigma_1^2\bar{\sigma}_2 - 4\sigma_2\bar{\sigma}_1^2$$
$$= 12\bar{\sigma}_1\sigma_1 - 4\sigma_1^2\bar{\sigma}_2 - 12\sigma_1\bar{\sigma}_1 + 4\sigma_2\bar{\sigma}_1^2 + 4\sigma_1^2\bar{\sigma}_2 - 4\sigma_2\bar{\sigma}_1^2 \equiv 0$$

That is, the carrier of the resultant passes through the endpoint of the directed line segment $\overrightarrow{4OH}$.

2°. Now suppose the orientations of the squares $\overrightarrow{A_1B_2C'C''}$, $\overrightarrow{B_1C_2A'A''}$, $\overrightarrow{C_1A_2B'B''}$ are the same but are opposite to the orientations of each of the squares $\overrightarrow{BCA_1A_2}$, $\overrightarrow{CAB_1B_2}$, $\overrightarrow{ABC_1C_2}$.

In order to keep Fig. 16 as simple as possible, only the centres P^*, Q^*, R^* of the squares $\overrightarrow{A_1B_2C'C''}$, $\overrightarrow{B_1C_2A'A''}$, $\overrightarrow{C_1A_2B'B''}$ are constructed; the points P^*, Q^*, R^* are symmetric to the corresponding points P, Q and R with respect to A_1B_2, B_1C_2, C_1A_2. The figure shows only three of the nine forces; namely the following forces are constructed: $\overrightarrow{P^*P_1^*}$, $\overrightarrow{P^*P_2^*}$, $\overrightarrow{P^*P_3^*}$, where P_1^*, P_2^*, P_3^* are the orthogonal projections of the point P^* on the sides BC, CA, AB.

The affix p^* of point P^* is

$$p^* = b_2 + \frac{1}{2}\overrightarrow{B_2A_1} + \frac{i}{2}\overrightarrow{B_2A_1}$$

$$= b_2 + \frac{1}{2}(a_1 - b_2) + \frac{i}{2}(a_1 - b_2) = \frac{1+i}{2}a_1 + \frac{1-i}{2}b_2$$

$$= \frac{1+i}{2}[(1+i)c - ib] + \frac{1-i}{2}[ia + (1-i)c] = \frac{1-i}{2}b + \frac{1+i}{2}a.$$

Similarly

$$q^* = \frac{1-i}{2}c + \frac{1+i}{2}b,$$

$$r^* = \frac{1-i}{2}a + \frac{1+i}{2}c,$$

where q^* and r^* are the affixes of points Q^* and R^*. Note that in this case as well,

$$p^* + q^* + r^* = a + b + c = \sigma_1.$$

Now, from the equations of the lines BC and $P^*P_1^*$,

$$z + bc\bar{z} = b + c,$$

$$z - bc\bar{z} = p^* - bc\bar{p}^*,$$

we find the affix p_1^* of point P_1^*:

$$p_1^* = \frac{1}{2}(b + c + p^* - bc\bar{p}^*)$$

and, hence,

$$\overrightarrow{P^*P_1^*} = p_1^* - p^* = \frac{1}{2}(b + c - p^* - bc\bar{p}^*).$$

Similarly,

$$\overrightarrow{P^*P_2^*} = \frac{1}{2}(c + a - p^* - ca\bar{p}^*),$$

$$\overrightarrow{P^*P_3^*} = \frac{1}{2}(a + b - p^* - ab\bar{p}^*)$$

and, consequently,

$$\overrightarrow{P^*P_1^*} + \overrightarrow{P^*P_2^*} + \overrightarrow{P^*P_3^*} = \sigma_1 - \frac{3}{2}p^* - \frac{1}{2}\sigma_2\bar{p}^*$$

and

$$\overrightarrow{Q^*Q_1^*} + \overrightarrow{Q^*Q_2^*} + \overrightarrow{Q^*Q_3^*} = \sigma_1 - \frac{3}{2}q^* - \frac{1}{2}\sigma_2\bar{q}^*,$$

$$\overrightarrow{R^*R_1^*} + \overrightarrow{R^*R_2^*} + \overrightarrow{R^*R_3^*} = \sigma_1 - \frac{3}{2}r^* - \frac{1}{2}\sigma_2\bar{r}^*,$$

and so the principal vector of the sum of nine forces $\overrightarrow{P^*P_1^*}, \overrightarrow{P^*P_2^*}, \ldots$ is equal to

$$\zeta = 3\sigma_1 - \frac{3}{2}(p^* + q^* + r^*) - \frac{1}{2}\sigma_2(\bar{p}^* + \bar{q}^* + \bar{r}^*)$$

$$= 3\sigma_1 - \frac{3}{2}\sigma_1 - \frac{1}{2}\sigma_2\bar{\sigma}_1 = \frac{3}{2}\sigma_1 - \frac{1}{2}\sigma_2\bar{\sigma}_1 = \frac{1}{2}(3\sigma_1 - \sigma_2\bar{\sigma}_1) = h',$$

where h' is the affix of the orthocentre of $\triangle A_hB_hC_h$.

As in the case of item 1°, let us now consider the sums of nine forces taken three at a time:

$$\overrightarrow{P^*P'^*} = \overrightarrow{P^*P_1^*} + \overrightarrow{P^*P_2^*} + \overrightarrow{P^*P_3^*},$$

$$\overrightarrow{Q^*Q'^*} = \overrightarrow{Q^*Q_1^*} + \overrightarrow{Q^*Q_2^*} + \overrightarrow{Q^*Q_3^*},$$

$$\overrightarrow{R^*R'^*} = \overrightarrow{R^*R_1^*} + \overrightarrow{R^*R_2^*} + \overrightarrow{R^*R_3^*}.$$

Let us find the affix p'^* of point P'^*:

$$\overrightarrow{P^*P'^*} = p'^* - p^* = \sigma_1 - \frac{3}{2}p^* - \frac{1}{2}\sigma_2\bar{p}^*,$$

whence we find p'^* and, similarly, q'^* and r'^*:

$$p'^* = \sigma_1 - \frac{1}{2}p^* - \frac{1}{2}\sigma_2\bar{p}^*,$$

$$q'^* = \sigma_1 - \frac{1}{2}q^* - \frac{1}{2}\sigma_2\bar{q}^*,$$

$$r'^* = \sigma_1 - \frac{1}{2}r^* - \frac{1}{2}\sigma_2\bar{r}^*.$$

The equation of the straight line carrying the resultant is of the form

$$\begin{vmatrix} z & p^* & p'^* \\ \bar{z} & \bar{p}^* & \bar{p}'^* \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} z & q^* & q'^* \\ \bar{z} & \bar{q}^* & \bar{q}'^* \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} z & r^* & r'^* \\ \bar{z} & \bar{r}^* & \bar{r}'^* \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

We have

$$\begin{vmatrix} z & p^* & p'^* \\ \bar z & \bar p^* & \bar p'^* \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} z & p^* & \sigma_1 - \frac{1}{2}p^* - \frac{1}{2}\sigma_2\bar p^* \\ \bar z & \bar p^* & \bar\sigma_1 - \frac{1}{2}\bar p^* - \frac{1}{2}\bar\sigma_2 p^* \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} z & p^* & \sigma_1 - \frac{1}{2}\sigma_2\bar p^* \\ \bar z & \bar p^* & \bar\sigma_1 - \frac{1}{2}\bar\sigma_2 p^* \\ 1 & 1 & \frac{3}{2} \end{vmatrix}$$

$$= \left(\frac{3}{2}\bar p^* - \bar\sigma_1 + \frac{1}{2}\bar\sigma_2 p^*\right) z + \left(\sigma_1 - \frac{1}{2}\sigma_2\bar p^* - \frac{3}{2}p^*\right)\bar z$$

$$+ p^*\bar\sigma_1 - \bar p^*\sigma_1 + \frac{1}{2}\sigma_2\bar p^{*2} - \frac{1}{2}\bar\sigma_2 p^{*2}.$$

If we write down two other similar expressions and then add them and equate the result to zero, we obtain the equation of that line in the case of item 2°;

$$\left(\frac{3}{2}\bar\sigma_1 - 3\bar\sigma_1 + \frac{1}{2}\bar\sigma_2\,\sigma_1\right) z + \left(3\sigma_1 - \frac{1}{2}\sigma_2\bar\sigma_1 - \frac{3}{2}\sigma_1\right)\bar z$$

$$+ \sigma_1\bar\sigma_1 - \sigma_1\bar\sigma_1 + \frac{1}{2}\sigma_2(\bar p^{*2} + \bar q^{*2} + \bar r^{*2}) - \frac{1}{2}\bar\sigma_2(p^{*2} + q^{*2} + r^{*2}) = 0$$

or

$$(3\bar\sigma_1 - \sigma_1\bar\sigma_2)\,z - (3\sigma_1 - \bar\sigma_1\sigma_2)\,\bar z$$
$$+ \bar\sigma_2(p^{*2} + q^{*2} + r^{*2}) - \sigma_2(\bar p^{*2} + \bar q^{*2} + \bar r^{*2}) = 0$$

Furthermore,

$$p^{*2} + q^{*2} + r^{*2} = \left(\frac{1-i}{2}b + \frac{1+i}{2}a\right)^2 + \left(\frac{1-i}{2}c + \frac{1+i}{2}b\right)^2$$

$$+ \left(\frac{1-i}{2}a + \frac{1+i}{2}c\right)^2 = -\frac{i}{2}(a^2 + b^2 + c^2) + ab + bc + ca$$

$$+ \frac{i}{2}(a^2 + b^2 + c^2) = \sigma_2,$$

$$\bar\sigma_2(p^{*2} + q^{*2} + r^{*2}) = \bar\sigma_2\sigma_2.$$

And therefore

$$\sigma_2(\bar p^{*2} + \bar q^{*2} + \bar r^{*2}) = \sigma_2\bar\sigma_2,$$

which means our equation takes the form

$$(3\bar\sigma_1 - \sigma_1\bar\sigma_2)\,z - (3\sigma_1 - \bar\sigma_1\sigma_2)\,\bar z = 0.$$

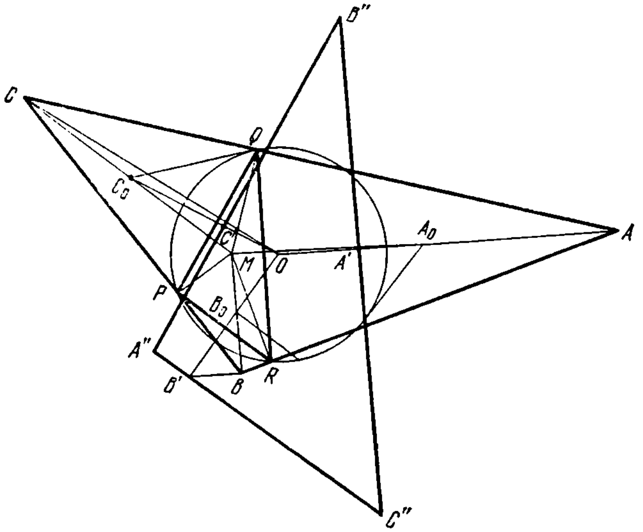

Fig. 17

Problem 14. Let P, Q, R be the orthogonal projections of point M on the sides BC, CA, AB of triangle ABC (Fig. 17). Denote by A', B', C' the points obtained by inversion of the midpoints A_0, B_0, C_0 of the segments MA, MB, MC with the circle of inversion (PQR) and by A'', B'', C'', the triangle formed by the polar lines of the points A_0, B_0, C_0 with respect to the same circle (PQR). Prove that

$$(A'B'C')^2 = \frac{1}{4}(A''B''C'')\cdot(PQR).$$

What is the necessary and sufficient condition for $\triangle\, \overrightarrow{A'B'C'}$ and $\triangle\, \overrightarrow{PQR}$ to have opposite orientations?

Solution. We take (PQR) for the unit circle. Let z_1, z_2, z_3 be the affixes of the points P, Q, R and let μ be the affix of point M.

The slope of the straight line MP is

$$\frac{\mu - z_1}{\bar{\mu} - \bar{z}_1}$$

and hence the slope of line BC is

$$-\frac{\mu - z_1}{\bar{\mu} - \bar{z}_1}$$

and the equation of line BC is of the form

$$z - z_1 = -\frac{\mu - z_1}{\bar{\mu} - \bar{z}_1}(\bar{z} - \bar{z}_1). \tag{40}$$

Similarly, the equation of line CA is

$$z - z_2 = -\frac{\mu - z_2}{\bar{\mu} - \bar{z}_2}(\bar{z} - \bar{z}_2). \tag{41}$$

Subtracting equation (41) from (40) term by term, we find the complex number $\bar{z} = \bar{c}$; it is the conjugate of the affix c of point C:

$$z_2 - z_1 = \left(\frac{\mu - z_2}{\bar{\mu} - \bar{z}_2} - \frac{\mu - z_1}{\bar{\mu} - \bar{z}_1}\right)\bar{c} + \bar{z}_1\frac{\mu - z_1}{\bar{\mu} - \bar{z}_1} - \bar{z}_2\frac{\mu - z_2}{\bar{\mu} - \bar{z}_2}$$

or

$$z_2 - z_1 = \frac{(z_2 - z_1)\left(-\bar{\mu} - \dfrac{\mu}{z_1 z_2} + \dfrac{z_1 + z_2}{z_1 z_2}\right)}{(\bar{\mu} - \bar{z}_1)(\bar{\mu} - \bar{z}_2)}\bar{c} + \frac{(z_2 - z_1)\dfrac{\mu\bar{\mu} - 1}{z_1 z_2}}{(\bar{\mu} - \bar{z}_1)(\bar{\mu} - \bar{z}_2)}.$$

Cancelling $z_2 - z_1$ and multiplying both sides by $(\bar{\mu} - \bar{z}_1)(\bar{\mu} - \bar{z}_2)$, we get

$$(\bar{\mu} - \bar{z}_1)(\bar{\mu} - \bar{z}_2) = \left(-\bar{\mu} - \frac{\mu}{z_1 z_2} + \frac{z_1 + z_2}{z_1 z_2}\right)\bar{c} + \frac{\mu\bar{\mu} - 1}{z_1 z_2},$$

whence

$$\left(-\bar{\mu} - \frac{\mu}{z_1 z_2} + \frac{z_1 + z_2}{z_1 z_2}\right)\bar{c} = (\bar{\mu} - \bar{z}_1)(\bar{\mu} - \bar{z}_2) - \bar{z}_1\bar{z}_2(\mu\bar{\mu} - 1)$$

or

$$(-\bar{\mu} - \mu\bar{z}_1\bar{z}_2 + \bar{z}_1 + \bar{z}_2)\bar{c} = \bar{\mu}^2 - \bar{\mu}(\bar{z}_1 + \bar{z}_2) + (2 - \mu\bar{\mu})\bar{z}_1\bar{z}_2.$$

This means that

$$\bar{c} = \frac{\bar{\mu}^2 - \bar{\mu}(\bar{z}_1 + \bar{z}_2) + (2 - \mu\bar{\mu})\bar{z}_1\bar{z}_2}{-\bar{\mu} - \mu\bar{z}_1\bar{z}_2 + \bar{z}_1 + \bar{z}_2}$$

and, hence,

$$c = \frac{\mu^2 - \mu(z_1 + z_2) + (2 - \mu\bar{\mu})z_1 z_2}{-\mu - \bar{\mu}z_1 z_2 + z_1 + z_2}$$

or

$$c = \frac{-\mu^2 + \mu(z_1 + z_2) + (\mu\bar{\mu} - 2)z_1 z_2}{\mu + \bar{\mu}z_1 z_2 - z_1 - z_2}$$

The affix c_0 of the midpoint of segment MC is

$$c_0 = \frac{1}{2}\left(\mu + \frac{-\mu^2 + \mu(z_1 + z_2) + (\mu\bar{\mu} - 2)z_1 z_2}{\mu + \bar{\mu}z_1 z_2 - (z_1 + z_2)}\right)$$

$$= \frac{(\mu\bar{\mu} - 1)z_1 z_2}{\mu + \bar{\mu}z_1 z_2 - z_1 - z_2},$$

and the affix c' of point C' obtained from C_0 by inversion with the circle of inversion (PQR) is

$$c' = \frac{1}{c_0} = \frac{\bar{\mu} + \mu\bar{z}_1\bar{z}_2 - \bar{z}_1 - \bar{z}_2}{(\mu\bar{\mu} - 1)\,\bar{z}_1\bar{z}_2} = \frac{\mu + \bar{\mu}z_1z_2 - z_1 - z_2}{\mu\bar{\mu} - 1}.$$

We find the affixes a' and b' of points A' and B' in similar fashion:

$$a' = \frac{\mu + \bar{\mu}z_2z_3 - z_2 - z_3}{\mu\bar{\mu} - 1},$$

$$b' = \frac{\mu + \bar{\mu}z_3z_1 - z_3 - z_1}{\mu\bar{\mu} - 1}.$$

We now find

$$(A'B'C') = \frac{i}{4}\begin{vmatrix} a' & \bar{a}' & 1 \\ b' & \bar{b}' & 1 \\ c' & \bar{c}' & 1 \end{vmatrix} = \frac{i}{4(\mu\bar{\mu}-1)^2}\begin{vmatrix} \mu + \bar{\mu}z_2z_3 - z_2 - z_3 & \bar{\mu} + \mu\bar{z}_2\bar{z}_3 - \bar{z}_2 - \bar{z}_3 & 1 \\ \mu + \bar{\mu}z_3z_1 - z_3 - z_1 & \bar{\mu} + \mu\bar{z}_3\bar{z}_1 - \bar{z}_3 - \bar{z}_1 & 1 \\ \mu + \bar{\mu}z_1z_2 - z_1 - z_2 & \bar{\mu} + \mu\bar{z}_1\bar{z}_2 - \bar{z}_1 - \bar{z}_2 & 1 \end{vmatrix}$$

$$= \frac{i}{4(\mu\bar{\mu}-1)^2}\begin{vmatrix} \bar{\mu}z_2z_3 - z_2 - z_3 & \mu\bar{z}_2\bar{z}_3 - \bar{z}_2 - \bar{z}_3 & 1 \\ \bar{\mu}z_3z_1 - z_3 - z_1 & \mu\bar{z}_3\bar{z}_1 - \bar{z}_3 - \bar{z}_1 & 1 \\ \bar{\mu}z_1z_2 - z_1 - z_2 & \mu\bar{z}_1\bar{z}_2 - \bar{z}_1 - \bar{z}_2 & 1 \end{vmatrix}$$

$$= \frac{i}{4(\mu\bar{\mu}-1)^2}\begin{vmatrix} \bar{\mu}\sigma_3\bar{z}_1 - \sigma_1 + z_1 & \mu\bar{\sigma}_3z_1 - \bar{\sigma}_1 + \bar{z}_1 & 1 \\ \bar{\mu}\sigma_3\bar{z}_2 - \sigma_1 + z_2 & \mu\bar{\sigma}_3z_2 - \bar{\sigma}_1 + \bar{z}_2 & 1 \\ \bar{\mu}\sigma_3\bar{z}_3 - \sigma_1 + z_3 & \mu\bar{\sigma}_3z_3 - \bar{\sigma}_1 + \bar{z}_3 & 1 \end{vmatrix}$$

$$= \frac{i}{4(\mu\bar{\mu}-1)^2}\begin{vmatrix} \bar{\mu}\sigma_3\bar{z}_1 + z_1 & \mu\bar{\sigma}_3z_1 + \bar{z}_1 & 1 \\ \bar{\mu}\sigma_3\bar{z}_2 + z_2 & \mu\bar{\sigma}_3z_2 + \bar{z}_2 & 1 \\ \bar{\mu}\sigma_3\bar{z}_3 + z_3 & \mu\bar{\sigma}_3z_3 + \bar{z}_3 & 1 \end{vmatrix}$$

$$= \frac{i}{4(\mu\bar{\mu} - 1)^2}\left[\mu\bar{\mu}\sigma_3\bar{\sigma}_3\begin{vmatrix} \bar{z}_1 & z_1 & 1 \\ \bar{z}_2 & z_2 & 1 \\ \bar{z}_3 & z_3 & 1 \end{vmatrix} + \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}\right]$$

$$= \frac{i(1 - \mu\bar{\mu})}{4(1 - \mu\bar{\mu})^2}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = \frac{(PQR)}{1 - \mu\bar{\mu}}.$$

Furthermore, the slope of the straight line OA_0 is

$$\varkappa_{OA_0} = \frac{(\mu\bar{\mu} - 1)\,z_2z_3}{\mu + \bar{\mu}z_2z_3 - z_2 - z_3} : \frac{(\mu\bar{\mu} - 1)\,\bar{z}_2\bar{z}_3}{\bar{\mu} + \mu\bar{z}_2\bar{z}_3 - \bar{z}_2 - \bar{z}_3}$$

$$= \frac{(\mu\bar{\mu} - 1)\,z_2z_3}{\mu + \bar{\mu}z_2z_3 - z_2 - z_3} : \frac{\mu\bar{\mu} - 1}{\mu + \bar{\mu}z_2z_3 - z_2 - z_3} = z_2z_3.$$

From this it is easy to find the equations of the polars of points A_0 and B_0 with respect to the circle (PQR); these are straight lines that pass through points A' and B' respectively and are perpendiculars to the lines OA' and OB';

$$z - \frac{\mu + \bar{\mu}z_2z_3 - z_2 - z_3}{\mu\bar{\mu} - 1} = - z_2z_3\left(\bar{z} - \frac{\bar{\mu} + \mu\bar{z}_2\bar{z}_3 - \bar{z}_2 - \bar{z}_3}{\mu\bar{\mu} - 1}\right)$$

and

$$z - \frac{\mu + \bar{\mu}z_3z_1 - z_3 - z_1}{\mu\bar{\mu} - 1} = - z_3z_1\left(\bar{z} - \frac{\bar{\mu} + \mu\bar{z}_3\bar{z}_1 - \bar{z}_3 - \bar{z}_1}{\mu\bar{\mu} - 1}\right).$$

The affix $z = c''$ of point C'', the point of intersection of these lines, can be found from the equation

$$-\frac{1}{z_2z_3}\left(z - \frac{\mu + \bar{\mu}z_2z_3 - z_2 - z_3}{\mu\bar{\mu} - 1}\right) + \frac{\bar{\mu} + \mu\bar{z}_2\bar{z}_3 - \bar{z}_2 - \bar{z}_3}{\mu\bar{\mu} - 1}$$
$$= -\frac{1}{z_3z_1}\left(z - \frac{\mu + \bar{\mu}z_3z_1 - z_3 - z_1}{\mu\bar{\mu} - 1}\right) + \frac{\bar{\mu} + \mu\bar{z}_3\bar{z}_1 - \bar{z}_3 - \bar{z}_1}{\mu\bar{\mu} - 1}$$

or

$$-\frac{z}{z_2z_3} + 2\frac{\mu + \bar{\mu}z_2z_3 - z_2 - z_3}{z_2z_3(\mu\bar{\mu} - 1)} = -\frac{z}{z_3z_1} + 2\frac{\mu + \bar{\mu}z_3z_1 - z_3 - z_1}{z_3z_1(\mu\bar{\mu} - 1)},$$

$$\frac{(z_2 - z_1)\,z}{\sigma_3} = \frac{2}{\mu\bar{\mu} - 1}\left[\frac{\mu}{z_3}\left(\frac{1}{z_1} - \frac{1}{z_2}\right) + \frac{1}{z_2} - \frac{1}{z_1}\right],$$

whence

$$c'' = \frac{2}{\mu\bar{\mu} - 1}(\mu - z_3).$$

Similarly,

$$a'' = \frac{2}{\mu\bar{\mu} - 1}(\mu - z_1)$$

and

$$b'' = \frac{2}{\mu\bar{\mu} - 1}(\mu - z_2)$$

so that

$$(A''B''C'') = \frac{i}{4}\frac{4}{(\mu\bar{\mu} - 1)^2}\begin{vmatrix} \mu - z_1 & \bar{\mu} - \bar{z}_1 & 1 \\ \mu - z_2 & \bar{\mu} - \bar{z}_2 & 1 \\ \mu - z_3 & \bar{\mu} - \bar{z}_3 & 1 \end{vmatrix}$$
$$= \frac{4}{(\mu\bar{\mu} - 1)^2}\cdot\frac{i}{4}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = \frac{4}{(\mu\bar{\mu} - 1)^2}(PQR).$$

Thus

$$(A'B'C') = \frac{(PQR)}{1-\mu\bar{\mu}}, \quad (A''B''C'') = \frac{4}{(1-\mu\bar{\mu})^2}(PQR),$$

whence

$$(A'B'C')^2 = \frac{(PQR)^2}{(1-\mu\bar{\mu})^2} = \frac{1}{4}(A''B''C'')\cdot(PQR).$$

The triangles $\overrightarrow{A'B'C'}$ and $\overrightarrow{PQR}$ have the same orientation if and only if $1-\mu\bar{\mu} > 0$, that is, the power of point M with respect to the circle (PQR) is negative; in other words, if and only if the point M lies *inside* (PQR). In Fig. 17, M lies inside (PQR) and, indeed, $\overrightarrow{A'B'C'}\downarrow\downarrow\overrightarrow{PQR}$.

Problem 15. Given $\triangle ABC$. Through an arbitrary point P of circle (ABC) circumscribed about $\triangle ABC$ are drawn lines parallel to the sides BC, CA, AB. Let A', B', C' be the respective second points of intersection of these lines with the circle (ABC). Denote by A'', B'', C'' the points symmetric to points A, B, C about the straight lines $B'C', C'A', A'B'$.

1°. Prove that $\triangle\overrightarrow{ABC}$ and $\triangle\overrightarrow{A''B''C''}$ are congruent but have opposite orientations.

2°. Prove that $OO'' = PH$, where O and O'' are centres of (ABC) and $(A''B''C'')$ and H is the orthocentre of $\triangle ABC$ (Fig. 18).

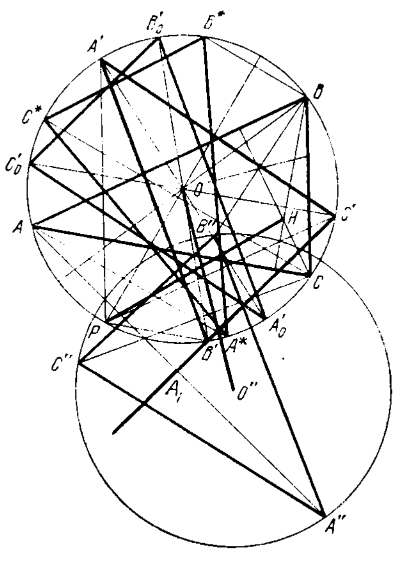

Fig. 18

Solution. Version One. 1°. Take (ABC) for the unit circle and assign to point P an affix of 1 (that is, let P be the unit point). Then the affixes a', b', c' of points A', B', C' are

$$a' = z_2 z_3, \quad b' = z_3 z_1, \quad c' = z_1 z_2,$$

where z_1, z_2, z_3 are the affixes of the vertices A, B, C of the triangle. The slope of line $B'C'$ is

$$\frac{z_1 z_2 - z_1 z_3}{\bar{z}_1\bar{z}_2 - \bar{z}_1\bar{z}_3} = \frac{z_1(z_2 - z_3)}{\frac{1}{z_1}\left(\frac{1}{z_2} - \frac{1}{z_3}\right)}$$

$$= -z_1^2 z_2 z_3 = -z_1\sigma_3$$

and so the equation of line $B'C'$ is of the form

$$z - z_1 z_3 = -z_1\sigma_3(\bar{z} - \bar{z}_1\bar{z}_3)$$

or

$$z + z_1\sigma_3\bar{z} = z_1 z_2 + z_1 z_3. \quad (42)$$

The equation of a perpendicular dropped from point A to $B'C'$ is

$$z - z_1 = z_1 \sigma_3(\bar{z} - \bar{z}_1)$$

or

$$z - z_1 \sigma_3 \bar{z} = z_1 - \sigma_3. \tag{43}$$

Combining equations (42) and (43) term by term, we find the affix $z = a_1$ of the orthogonal projection A_1 of point A on the line $B'C'$:

$$a_1 = \frac{1}{2}(z_1 + z_1 z_2 + z_1 z_3 - \sigma_3) = \frac{1}{2}(z_1 - z_2 z_3 + \sigma_2 - \sigma_3).$$

The affix a'' of point A'', symmetric to A with respect to $B'C'$, is found from the relation

$$\frac{z_1 + a''}{2} = a_1,$$

whence

$$a'' = 2a_1 - z_1 = \sigma_2 - \sigma_3 - z_2 z_3.$$

In similar fashion we find the affixes b'' and c'' of points B'' and C'':

$$b'' = \sigma_2 - \sigma_3 - z_3 z_1, \qquad c'' = \sigma_2 - \sigma_3 - z_1 z_2.$$

From these relations it follows that the centre O'' of circle $(A''B''C'')$ has the affix

$$o'' = \sigma_2 - \sigma_3,$$

and the radius is equal to 1 since

$$|a'' - o''| = |b'' - o''| = |c'' - o''| = 1$$

$$(|-z_2 z_3| = |-z_3 z_1| = |-z_1 z_2| = 1).$$

Let us consider $\triangle \overrightarrow{A_0'B_0'C_0'}$ the affixes of the vertices of which are equal respectively to $-z_2 z_3$, $-z_3 z_1$, $-z_1 z_2$. The triangle $\overrightarrow{A_0'B_0'C_0'}$ is symmetric to $\triangle \overrightarrow{A'B'C'}$ with respect to the point O, and, hence, the triangle $\overrightarrow{A''B''C''}$ is obtained by a translation of $\triangle \overrightarrow{A_0'B_0'C_0'}$ via the directed line segment $\overrightarrow{OT}$, where the affix τ of point T is equal to $\sigma_2 - \sigma_3$.

Furthermore,

$$-z_2 z_3 = -\frac{\sigma_3}{z_1} = -\sigma_3 \bar{z}_1, \quad -z_3 z_1 = -\sigma_3 \bar{z}_2, \quad -z_1 z_2 = -\sigma_3 \bar{z}_3.$$

Therefore $\triangle \overrightarrow{A_0'B_0'C_0'}$ is obtained from $\triangle \overrightarrow{A^*B^*C^*}$ (the affixes of the vertices of which are $\bar{z}_1$, $\bar{z}_2$, $\bar{z}_3$) by a rotation about O through the angle $\arg(-\sigma_3)$.

The triangle $\overrightarrow{A^*B^*C^*}$ is symmetric to $\triangle\overrightarrow{ABC}$ about line OP (the real axis Ox), and therefore $\triangle\overrightarrow{ABC}$ and $\triangle\overrightarrow{A^*B^*C^*}$ are equal and have opposite orientations. But $\triangle\overrightarrow{A^*B^*C^*}$ is equal to $\triangle\overrightarrow{A_0'B_0'C_0'}$ and has the same orientation; $\triangle\overrightarrow{A_0'B_0'C_0'}$ is equal to $\triangle\overrightarrow{A'B'C'}$ and has the same orientation, while $\triangle\overrightarrow{A''B''C''}$ is equal to $\triangle\overrightarrow{A_0'B_0'C_0'}$ and has the same orientation. Consequently, $\overrightarrow{ABC} \downarrow\uparrow \overrightarrow{A''B''C''}$ and $\triangle ABC = \triangle A''B''C''$ (the symbol $=$ here signifies congruence).

2°. The affix of point O'' is equal to $o'' = \sigma_2 - \sigma_3$, whence

$$OO'' = |\sigma_2 - \sigma_3| = |\sigma_3| \left| \frac{\sigma_2}{\sigma_3} - 1 \right| = \left| \frac{\sigma_2}{\sigma_3} - 1 \right| = |\bar{\sigma}_1 - 1| = |\sigma_1 - 1| = PH.$$

Version Two. 1°. Take (ABC) for the unit circle. Let z_1, z_2, z_3, p be the affixes of points A, B, C, P respectively. Take the Boutain point of $\triangle ABC$ for the unit point so that $\sigma_3 = 1$. The affixes a', b', c' of points A', B', C' are

$$a' = \frac{z_2 z_3}{p}, \quad b' = \frac{z_3 z_1}{p} \quad c' = \frac{z_1 z_2}{p}$$

or $(\sigma_3 = 1)$

$$a' = \bar{p}\,\bar{z}_1, \quad b' = \bar{p}\,\bar{z}_2, \quad c' = \bar{p}\,\bar{z}_3.$$

The slope of line $B'C'$ is

$$\frac{b' - c'}{\bar{b}' - \bar{c}'} = \frac{\bar{p}\,\bar{z}_2 - \bar{p}\,\bar{z}_3}{pz_2 - pz_3} = -\frac{1}{z_2 z_3 p^2} = -\frac{z_1}{p_2} \quad (\sigma_3 = 1)$$

and the equation of $B'C'$ is of the form

$$z - \bar{p}\,\bar{z}_2 = -z_1\,\bar{p}^2(\bar{z} - pz_2)$$

or

$$z + z_1\,\bar{p}^2\,\bar{z} = \bar{p}\,\bar{z}_2 + \bar{p}\,\bar{z}_3. \tag{44}$$

The equation of the perpendicular dropped from point A on line $B'C'$ is

$$z - z_1 = z_1\,\bar{p}^2(\bar{z} - \bar{z}_1)$$

or

$$z - z_1\,\bar{p}^2\,\bar{z} = z_1 - \bar{p}^2. \tag{45}$$

Combining the equations (44) and (45) term by term, we find the affix $z = a_1$ of projection $A_\perp$ of point A on the straight line $B'C'$:

$$a_1 = \frac{1}{2}(z_1 + \bar{p}\,\bar{z}_2 + \bar{p}\,\bar{z}_3 - \bar{p}^2).$$

The affix a'' of point A'', symmetric to point A about $B'C'$, is found from the relation

$$\frac{z_1 + a''}{2} = a_1$$

whence

$$a'' = 2a_1 - z_1 = \bar{p}\,\bar{z}_2 + \bar{p}\,\bar{z}_3 - \bar{p}^2.$$

Similarly,

$$b'' = \bar{p}\,\bar{z}_3 + \bar{p}\,\bar{z}_1 - \bar{p}^2.$$

We now find

$$a'' - b'' = \bar{p}(\bar{z}_2 - \bar{z}_1),$$
$$\bar{a}'' - \bar{b}'' = p(z_2 - z_1)$$

and, hence,

$$A''B''^2 = (a'' - b'')(\bar{a}'' - \bar{b}'') = p\bar{p}(z_2 - z_1)(\bar{z}_2 - \bar{z}_1) = AB^2.$$

That is, $A''B'' = AB$. In similar manner it can be proved that $B''C'' = BC$, $C''A'' = CA$, that is $\triangle ABC$ and $\triangle A''B''C''$ are congruent.

In order to prove that $\triangle \overrightarrow{ABC}$ and $\Delta \overrightarrow{A''B''C''}$ have opposite orientations, it suffices to prove that

$$\Delta = \begin{vmatrix} z_1 & \bar{a}'' & 1 \\ z_2 & \bar{b}'' & 1 \\ z_3 & \bar{c}'' & 1 \end{vmatrix} = 0.$$

We have

$$\Delta = \begin{vmatrix} z_1 & p(z_2 + z_3) - p^2 & 1 \\ z_2 & p(z_3 + z_1) - p^2 & 1 \\ z_3 & p(z_1 + z_3) - p^2 & 1 \end{vmatrix} = \begin{vmatrix} z_1 & p(\sigma_1 - z_1) & 1 \\ z_2 & p(\sigma_1 - z_2) & 1 \\ z_3 & p(\sigma_1 - z_3) & 1 \end{vmatrix} = \begin{vmatrix} z_1 & -pz_1 & 1 \\ z_2 & -pz_2 & 1 \\ z_3 & -pz_3 & 1 \end{vmatrix} = 0$$

and so $\triangle ABC = \triangle A''B''C''$ and $\overrightarrow{ABC} \downarrow\uparrow \overrightarrow{A''B''C''}$.

2°. From the formulas

$$a'' = \bar{p}\,\bar{z}_2 + \bar{p}\,\bar{z}_3 - \bar{p}^2 = \bar{p}(\bar{\sigma}_1 - \bar{z}\bar{\sigma}_1) - \bar{p}^2 = -\bar{p}\,\bar{z}_1 + \bar{p}\bar{\sigma}_1 - \bar{p}^2,$$
$$b'' = -\bar{p}\,\bar{z}_2 + \bar{p}\bar{\sigma}_1 - \bar{p}^2,$$
$$c'' = -\bar{p}\,\bar{z}_3 + \bar{p}\bar{\sigma}_1 - \bar{p}^2$$

it follows that the affix o'' of the centre O'' of the circle $(A''B''C'')$ is

$$o'' = \bar{p}\bar{\sigma}_1 - \bar{p}^2$$

and the radius is equal to 1 ($|-\bar{p}\,\bar{z}_1| = |-\bar{p}\,\bar{z}_2| = |-\bar{p}\,\bar{z}_3| = 1$), whence

$$OO'' = |o''| = |\bar{p}\bar{\sigma}_1 - \bar{p}^2| = |\bar{p}|\,|\bar{\sigma}_1 - \bar{p}| = |\bar{\sigma}_1 - \bar{p}| = |\sigma_1 - p| = PH.$$

Remark. From the relation $o'' = \bar{p}(\bar{\sigma}_1 - \bar{p})$ it follows that the directed line segment $\overrightarrow{OO''}$ is equivalent to the directed line segment obtained by symmetry in the x-axis of the directed line segment $\overrightarrow{PH}$ and a subsequent rotation through the angle $\arg \bar{p}$ (the x-axis is a straight line passing through point O and the Boutain point).

Problem 16. On the sides BC, CA, AB are constructed triangles $\overrightarrow{A'BC}$, $\overrightarrow{B'CA}$, $\overrightarrow{C'AB}$, which are similar and have the same orientation. Let P be an arbitrary point lying on the circle $(O) = (ABC)$. The directed line segments $\overrightarrow{OA'}, \overrightarrow{OB'}, \overrightarrow{OC'}$ rotate about point O through angles that are respectively equal to $(\overrightarrow{OP}, \overrightarrow{OA})$, $(\overrightarrow{OP}, \overrightarrow{OB})$, $(\overrightarrow{OP}, \overrightarrow{OC})$ (these angles are oriented). Let $\overrightarrow{OA''}, \overrightarrow{OB''}, \overrightarrow{OC''}$ be the respectively rotated segments. Prove that the centroid G'' of $\triangle A''B''C''$ is symmetric to the centroid G of $\triangle ABC$ about the diameter of the circle (O), which diameter is parallel to the Simson line constructed for point P with respect to $\triangle ABC$.

Solution. We take the circle $(O) = (ABC)$ for the unit circle, and the point P for the unit point. Let z_1, z_2, z_3 be the affixes of the points A, B, C. Denote by a', b', c' the affixes of the points A', B', C'. Since the triangles $\overrightarrow{A'BC}$, $\overrightarrow{B'CA}$, $\overrightarrow{C'AB}$ are similar and have the same orientation, it follows, assuming

$$\varphi = (\overrightarrow{CB}, \overrightarrow{CA'}), \qquad \rho = \frac{CA'}{CB},$$

we will have

$$a' = z_3 + \rho(\cos\varphi + i\sin\varphi)(z_2 - z_3) = \alpha z_2 + (1-\alpha) z_3,$$

where $\alpha = \rho(\cos\varphi + i\sin\varphi)$, and similar expressions for b' and c' with the same value of α:

$$b' = \alpha z_3 + (1-\alpha) z_1,$$

$$c' = \alpha z_1 + (1-\alpha) z_2.$$

The affixes a'', b'', c'' of the points A'', B'', C'' are

$$a'' = z_1 a', \quad b'' = z_2 b', \quad c'' = z_3 c'.$$

That is,

$$a'' = z_1 z_2 \alpha + (1-\alpha) z_1 z_3,$$

$$b'' = z_2 z_3 \alpha + (1-\alpha) z_2 z_1,$$

$$c'' = z_3 z_1 \alpha + (1-\alpha) z_3 z_2.$$

From this we find the affix g'' of the centroid of $\triangle A''B''C''$:

$$g'' = \frac{a'' + b'' + c''}{3} = \frac{\sigma_2}{3}$$

where $\sigma_2 = z_1 z_2 + z_2 z_3 + z_3 z_1$.

The equation of the diameter of (O), which diameter is parallel to the Simson line constructed for the unit point P with respect to $\triangle ABC$, is of the form (see problem 3)

$$z - \sigma_3 \bar{z} = 0.$$

The ends of that diameter have affixes $\sqrt{\sigma_3}$ ($\sqrt{\sigma_3}$ has two values; each of them satisfies the equation $z - \sigma_3 \bar{z} = 0$).

In order to prove that the points G and G'' are symmetric about the diameter $z - \sigma_3 \bar{z} = 0$, it suffices to prove that $\triangle ODG$ and $\triangle ODG''$ are similar but have opposite orientations (D is one of the ends of the diameter $z - \sigma_3 \bar{z} = 0$), that is, that

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ g & \bar{g}'' & 1 \\ \sqrt{\sigma_3} & \overline{\sqrt{\sigma_3}} & 1 \end{vmatrix} = 0$$

(for $\sqrt{\sigma_3}$, any one of the two values may be taken; $\overline{\sqrt{\sigma_3}}$ is the conjugate of that value). We then have

$$\Delta = \overline{\sqrt{\sigma_3}}\, g - \sqrt{\sigma_3}\, \bar{g}'' = \overline{\sqrt{\sigma_3}}\,(g - \sigma_3 \bar{g}'')$$

$$= \overline{\sqrt{\sigma_3}} \left(\frac{\sigma_1}{3} - \sigma_3 \frac{\bar{\sigma}_2}{3} \right) = \frac{1}{3} \overline{\sqrt{\sigma_3}} \left(\sigma_1 - \sigma_3 \frac{\sigma_1}{\sigma_3} \right) = 0.$$

Problem 17. The altitudes of an arbitrary triangle ABC intersect the circle $(O) = (ABC)$ in the points A_1, B_1, C_1; A', B', C' are points symmetric to point P, which lies on (O), with respect to the straight lines OA, OB, OC; A'', B'', C'' are points symmetric to the point P with respect to the lines OA', OB', OC'; α, β, γ are points symmetric to the points A'', B'', C'' about the line OP. Prove that the points A_2, B_2, C_2, which are symmetric to the points α, β, γ with respect to the tangents to (O) at the points A_1, B_1, C_1, form a triangle $A_2B_2C_2$ that is homothetic to $\triangle A_1B_1C_1$ with ratio equal to 2; the centre Q of this homothetic transformation belongs to the circle (O).

Solution. Take $(O) = (ABC)$ for the unit circle and P for the unit point of the complex-variable plane. Let z_1, z_2, z_3 be the affixes of the points A, B, C.

The equation of the straight line BC is of the form

$$z + z_2 z_3 \bar{z} = z_2 + z_3.$$

The equation of an altitude dropped from the vertex A on the line BC is

$$z - z_1 = z_2 z_3(\bar{z} - \bar{z}_1).$$

Solving this equation together with the equation $z\bar{z} = 1$ of the unit circle, we get

$$z - z_1 = z_2 z_3\left(\frac{1}{z} - \frac{1}{z_1}\right) \quad \text{or} \quad z - z_1 = -z_2 z_3 \frac{z - z_1}{z_1 z}.$$

One of the roots of this equation is $z = z_1$ (the affix of point A); the other is

$$z = a_1 = -\frac{z_2 z_3}{z_1}$$

(the affix of point A_1). Similarly,

$$b_1 = -\frac{z_3 z_1}{z_2}, \qquad c_1 = -\frac{z_1 z_2}{z_3}.$$

The equation of the line OA is

$$z - z_1^2 \bar{z} = 0$$

and the equation of a perpendicular dropped from point P on the same line is

$$z - 1 = -z_1^2(\bar{z} - 1).$$

Solve this equation together with the equation $z\bar{z} = 1$ of the unit circle:

$$z - 1 = -z_1^2\left(\frac{1}{z} - 1\right), \quad \text{that is,} \quad z - 1 = z_1^2 \frac{z - 1}{z}.$$

One of the roots of this equation is $z = 1$ (the affix of point P), the other is the affix a' of point A':

$$z = a' = z_1^2.$$

Similarly,

$$b' = z_2^2, \qquad c' = z_3^2.$$

are the affixes of points B' and C'.

The equation of the straight line OA' is

$$z - z_1^4 \bar{z} = 0$$

and the equation of the straight line passing through point P perpendicularly to line OA' is

$$z - 1 = -z_1^4(\bar{z} - 1).$$

From this and from the equation $z\bar{z} = 1$ of the unit circle we find the affix a'' of point A'':

$$z - 1 = -z_1^4\left(\frac{1}{z} - 1\right), \quad \text{that is,} \quad z - 1 = z_1^4 \frac{z-1}{z}.$$

One of the roots of this equation is $z = 1$ (the affix of point P), the other is

$$z = a'' = z_1^4.$$

Similarly,

$$b'' = z_2^4, \qquad c'' = z_3^4,$$

where b'' and c'' are the affixes of the points B'' and C''.

The slope of the straight line OP is equal to 1. Hence, the equation of the straight line passing through point A'' perpendicularly to line OP is of the form

$$z - a'' = -(\bar{z} - \bar{a}'').$$

Solving this equation together with the equation $z\bar{z} = 1$ of the unit circle, we find the affix λ of point α:

$$z - a'' = -\left(\frac{1}{z} - \frac{1}{a''}\right)$$

or

$$z - a'' = \frac{z - a''}{a''z}.$$

One of the roots of this equation is $z = a''$ (the affix of point A''); the other is

$$z = \lambda = \frac{1}{a''} = \frac{1}{z_1^4},$$

which is the affix of point α. In similar fashion we find the affixes μ and ν of the points β and γ:

$$\mu = \frac{1}{z_2^4}, \qquad \nu = \frac{1}{z_3^4}.$$

The slope of line OA_1 is

$$\frac{a_1}{\bar{a}_1} = a_1^2 = \frac{z_2^2 z_3^2}{z_1^2}$$

and so the equation of the tangent line to the unit circle (O) at the point A_1 is of the form

$$z + \frac{z_2 z_3}{z_1} = -\frac{z_2^2 z_3^2}{z_1^2}\left(\bar{z} + \frac{z_1}{z_2 z_3}\right)$$

or

$$z + \frac{z_2^2 z_3^2}{z_1^2} \bar{z} = -2 \frac{z_2 z_3}{z_1}. \tag{46}$$

The equation of the perpendicular dropped from the point α on this tangent line is of the form

$$z - \frac{1}{z_1^4} = \frac{z_2^2 z_3^2}{z_1^2} (\bar{z} - z_1^4)$$

or

$$z - \frac{z_2^2 z_3^2}{z_1^2} \bar{z} = \frac{1}{z_1^4} - \sigma_3^2. \tag{47}$$

Combining equations (46) and (47) term by term, we find the affix a_2^* of the projection of point α on the tangent line:

$$z = a_2^* = -\frac{z_2 z_3}{z_1} - \frac{\sigma_3^2}{2} + \frac{1}{2z_1^4}.$$

The affix a_2 of point A_2, which is symmetric to point α with respect to the tangent to the unit circle at point A_1, is found from the relation

$$\frac{\lambda_1 + a_2}{2} = a_2^*,$$

whence

$$a_2 = 2a_2^* - \lambda_1 = -2 \frac{z_2 z_3}{z_1} - \sigma_3^2 + \frac{1}{z_1^4} - \frac{1}{z_1^4} = -2 \frac{z_2 z_3}{z_1} - \sigma_3^2.$$

In similar fashion we find the affixes b_2 and c_2 of points B_2 and C_2:

$$b_2 = -2 \frac{z_3 z_1}{z_2} - \sigma_3^2, \qquad c_2 = -2 \frac{z_1 z_2}{z_3} - \sigma_3^2.$$

From these relations it follows that the straight line $A_1 A_2$ passes through point Q with affix

$$q = \sigma_3^2$$

since the midpoint of segment $A_2 Q$ has the affix

$$\frac{a_2 + q}{2} = -\frac{z_3 z_2}{z_1} = a_1,$$

which is to say the midpoint of $A_2 Q$ coincides with point A_1. Point Q lies on the unit circle since $|q| = 1$.

In similar fashion, we can prove that the points B_1 and C_1 are, respectively, the midpoints of line segments B_2Q and C_2Q. Thus,

$$\frac{\overrightarrow{QA_2}}{\overrightarrow{QA_1}}=\frac{\overrightarrow{QB_2}}{\overrightarrow{QB_1}}=\frac{\overrightarrow{QC_2}}{\overrightarrow{QC_1}}=2.$$

That is, $\triangle A_2B_2C_2$ is an image of $\triangle A_1B_1C_1$ under a homothetic transformation with centre Q lying on (ABC) and with homothetic ratio 2.

Problem 18. 1°. Through the vertices A_1, A_2, A_3 of $\triangle A_1A_2A_3$ lying on an oriented plane, draw parallel lines intersecting the given straight line Δ at the points P_1, P_2, P_3; note that the angle *from* the line Δ *to* the lines A_1P_1, A_2P_2, A_3P_3 is equal to α (Fig. 19).

Through points P_1, P_2, P_3 draw straight lines l_1, l_2, l_3 that intersect the respective sides A_2A_3, A_3A_1, A_1A_2; note that angles reckoned *from* the straight lines A_2A_3, A_3A_1, A_1A_2 *to* the straight lines l_1, l_2, l_3 are all equal to β. Prove that the lines l_1, l_2, l_3 form a triangle $Q_1Q_2Q_3$, which is similar to the triangle $A_1A_2A_3$; the factor of proportionality is

$$\left|\frac{\sin(\alpha+\beta)}{\sin\alpha}\right|.$$

Consider the following special cases;

2°. $\beta=\pi-\alpha$.

3°. $\beta=0$.

Solution. 1°. Take the circle $(O)=(A_1A_2A_3)$ for the unit circle. Let z_1, z_2, z_3 be the affixes of the points A_1, A_2, A_3 and let

$$az+\bar{a}\bar{z}=b$$

be the equation of the straight line Δ ($a\neq 0$ and b is a real number). Put

$$\lambda=\cos 2\alpha+i\sin 2\alpha,\quad \mu=\cos 2\beta+i\sin 2\beta.$$

The equation of line A_1P_1 may be written in the form

$$a(z-z_1)+\lambda\bar{a}(\bar{z}-\bar{z}_1)=0. \tag{48}$$

Indeed, the slope of the straight line Δ is

$$\varkappa=-\frac{\bar{a}}{a}$$

and the slope of the line (48) is

$$\varkappa'=-\frac{\lambda\bar{a}}{a}.$$

Hence

$$\frac{\varkappa'}{\varkappa}=\lambda$$

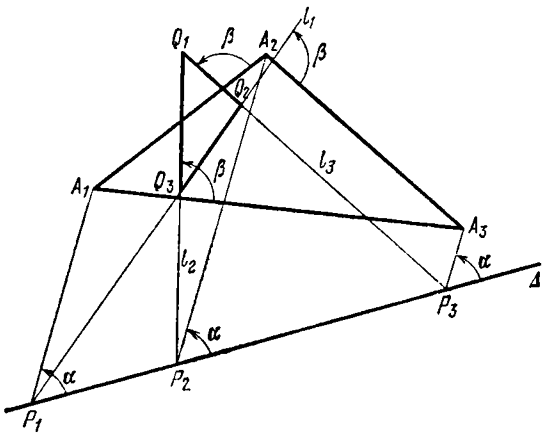

Fig. 19

and so

$$\sqrt{\frac{\varkappa'}{\varkappa}} = \sqrt{\lambda} = \cos\alpha + i\sin\alpha.$$

That is, the angle from the straight line Δ to the straight line (48) is equal to α.

If we took the other value of $\sqrt{\lambda}$,

$$\sqrt{\lambda} = -(\cos\alpha + i\sin\alpha) = \cos(\alpha+\pi) + i\sin(\alpha+\pi),$$

then the angle from the straight line Δ to the straight line (48) would turn out equal to $\pi+\alpha$, that is, it would be congruent to α modulo π. From the system of equations of the straight lines Δ and A_1P_1, that is, from the system of equations

$$az + \bar{a}\bar{z} = b,$$

$$az + \lambda\bar{a}\bar{z} = az_1 + \lambda\bar{a}\bar{z}_1,$$

we find the affix p_1 of point P_1:

$$z = p_1 = \frac{1}{1-\lambda}\left(z_1 - \frac{\lambda b}{a} + \lambda\frac{\bar{a}\bar{z}_1}{a}\right),$$

whence

$$\bar{p}_1 = \frac{1}{1-\bar{\lambda}}\left(\bar{z}_1 - \frac{\bar{\lambda}b}{\bar{a}} + \bar{\lambda}\frac{az_1}{\bar{a}}\right)$$

$$= \frac{\lambda}{\lambda-1}\left(\frac{1}{z_1} - \frac{b}{\lambda\bar{a}} + \frac{a\bar{z}_1}{\lambda\bar{a}}\right) = \frac{\lambda}{1-\lambda}\left(-\frac{1}{z_1} + \frac{b}{\lambda\bar{a}} - \frac{az_1}{\bar{a}}\right).$$

The affixes p_2 and p_3 of points P_2 and P_3 have similar expressions:

$$p_2 = \frac{1}{1-\lambda}\left(z_2 - \frac{\lambda b}{a} + \lambda\frac{\bar{a}\,\bar{z}_2}{a}\right),$$

$$p_3 = \frac{1}{1-\lambda}\left(z_3 - \frac{\lambda b}{a} + \lambda\frac{\bar{a}\,\bar{z}_3}{a}\right).$$

Furthermore, the equation of line A_2A_3 is of the form

$$z - z_2 = -z_2 z_3(\bar{z} - \bar{z}_2)$$

and so the equation of line l_1 may be written in the form

$$z - p_1 = -\mu z_2 z_3(\bar{z} - \bar{p}_1)$$

or

$$z - \frac{1}{1-\lambda}\left(z_1 - \frac{\lambda b}{a} + \frac{\lambda\bar{a}}{az_1}\right) = -\mu z_2 z_3\left[\bar{z} - \frac{\lambda}{1-\lambda}\left(-\frac{1}{z_1} + \frac{b}{\lambda\bar{a}} - \frac{az_1}{\lambda\bar{a}}\right)\right]$$

or

$$(1-\lambda)(z + \mu z_2 z_3\bar{z}) = z_1 - \frac{\lambda b}{a} + \frac{\lambda\bar{a}}{az_1} - \frac{\lambda\mu\, z_2 z_3}{z_1} + \frac{\mu b z_2 z_3}{\bar{a}} - \frac{\mu a\,\sigma_3}{\bar{a}}. \quad (49)$$

The equations of the straight lines l_2 and l_3 are written down in similar fashion:

$$(1-\lambda)(z + \mu z_3 z_1\bar{z}) = z_2 - \frac{\lambda b}{a} + \frac{\lambda\bar{a}}{az_2} - \frac{\lambda\mu\, z_3 z_1}{z_2} + \frac{\mu b\, z_3 z_1}{\bar{a}} - \frac{\mu a\,\sigma_3}{\bar{a}}, \quad (50)$$

$$(1-\lambda)(z + \mu z_1 z_2\bar{z}) = z_3 - \frac{\lambda b}{a} + \frac{\lambda\bar{a}}{az_3} - \frac{\lambda\mu\, z_1 z_2}{z_3} + \frac{\mu b\, z_1 z_2}{\bar{a}} - \frac{\mu a\,\sigma_3}{\bar{a}}. \quad (51)$$

Multiplying both sides of equation (49) by $-z_1$, both sides of (50) by z_2, and adding termwise, we obtain

$$(1-\lambda)(z_2 - z_1)z$$
$$= z_2^2 - z_1^2 - \frac{\lambda b}{a}(z_2 - z_1) + z_3(z_2 - z_1)\lambda\mu - \mu\frac{a}{\bar{a}}(z_2 - z_1)\sigma_3,$$

whence we get the affix q_3 of point Q_3, the point of intersection of the lines l_1 and l_2:

$$q_3 = \frac{z_1 + z_2 - \dfrac{\lambda b}{a} + \lambda\mu\, z_3 - \mu\dfrac{a}{\bar{a}}\sigma_3}{1-\lambda}$$

$$= \frac{\sigma_1 - \dfrac{\lambda b}{a} - \mu\dfrac{a}{\bar{a}}\sigma_3 + (\lambda\mu - 1)z_3}{1-\lambda}.$$

The affixes q_1 and q_2 of the points Q_1 and Q_2 have similar expressions:

$$q_1 = \frac{\sigma_1 - \frac{\lambda b}{a} - \mu \frac{a}{\bar{a}} \sigma_3 + (\lambda\mu - 1) z_1}{1 - \lambda},$$

$$q_2 = \frac{\sigma_1 - \frac{\lambda b}{a} - \mu \frac{a}{\bar{a}} \sigma_3 + (\lambda\mu - 1) z_2}{1 - \lambda}.$$

From the last three relations it follows that the points Q_1, Q_2, Q_3 are obtained from the points A_1, A_2, A_3 by a linear transformation of the first kind:

$$q = mz + n,$$

where

$$m = \frac{\lambda\mu - 1}{1 - \lambda},$$

$$n = \frac{\sigma_1 - \frac{\lambda b}{a} - \mu \frac{a}{\bar{a}} \sigma_3}{1 - \lambda},$$

and so $\triangle \overrightarrow{A_1A_2A_3}$ and $\triangle \overrightarrow{Q_1Q_2Q_3}$ are similar and have the same orientation (see Fig. 19): n is the affix of point O', into which the point O passes under the transformation; in other words, $(O') = (Q_1Q_2Q_3)$. The factor of proportionality is

$$|m| = \left| \frac{\lambda\mu - 1}{1 - \lambda} \right| = \frac{|\cos(2\alpha + 2\beta) + i \sin(2\alpha + 2\beta) - 1|}{|1 - \cos 2\alpha - i \sin 2\alpha|}$$

$$= \frac{|2 \sin^2(\alpha + \beta) - 2i \sin(\alpha + \beta) \cos(\alpha + \beta)|}{|2 \sin^2 \alpha - 2i \sin \alpha \cos \alpha|}$$

$$= \frac{|\sin(\alpha + \beta)|}{|\sin \alpha|} \frac{|\sin(\alpha + \beta) - i \cos(\alpha + \beta)|}{|\sin \alpha - i \cos \alpha|} = \left| \frac{\sin(\alpha + \beta)}{\sin \alpha} \right|.$$

2°. If $\beta = \pi - \alpha$ (or $\beta = -\alpha$), then $\triangle Q_1Q_2Q_3$ degenerates into a point. We have the following theorem: if through the vertices A_1, A_2, A_3 of $\triangle A_1A_2A_3$ lying on an oriented plane we draw parallel lines intersecting the given straight line Δ at points P_1, P_2, P_3, the angle from the straight line Δ to the straight lines A_1P_1, A_2P_2, A_3P_3 being equal to α, and then through the points P_1, P_2, P_3 we draw straight lines l_1, l_2, l_3 cutting the sides A_2A_3, A_3A_1, A_1A_2, the angles from the straight lines A_2A_3, A_3A_1, A_1A_2 to the straight lines l_1, l_2, l_3 respectively being equal to α, then the lines l_1, l_2, l_3 pass through one point (Fig. 20).

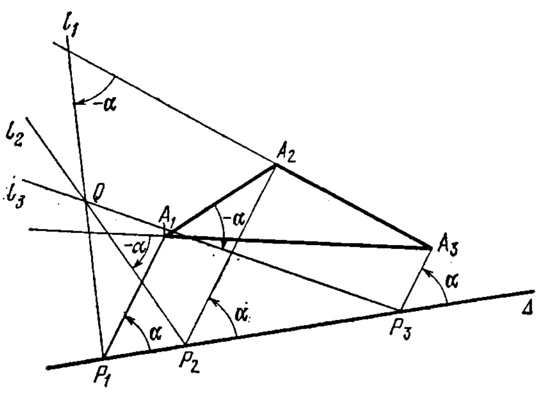

Fig. 20

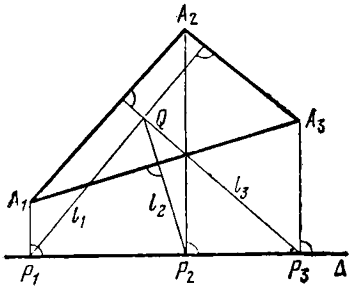

Fig. 21

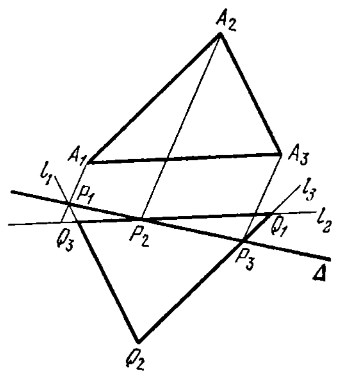

Fig. 22

In particular, if $\alpha = \pi/2$, $\beta = \pi/2$, we obtain the *theorem on the orthopole* of a straight line with respect to a triangle: if P_1, P_2, P_3 are the orthogonal projections of the vertices A_1, A_2, A_3 of $\triangle A_1A_2A_3$ on a given straight line Δ, then the lines that pass through the points P_1, P_2, P_3 and that are perpendicular to the straight lines A_2A_3, A_3A_1, A_1A_2 respectively, intersect in one point Q (the so-called *orthopole of the straight line* Δ with respect to $\triangle A_1A_2A_3$; Fig. 21).

3°. Suppose $\beta = 0$. Then we obtain the following theorem: if through the vertices A_1, A_2, A_3 of $\triangle A_1A_2A_3$ we draw parallel straight lines intersecting the given line Δ in the points P_1, P_2, P_3, and then draw through the points P_1, P_2, P_3 lines l_1, l_2, l_3 respectively parallel to lines A_2A_3, A_3A_1, A_1A_2, then the lines l_1, l_2, l_3 form a triangle $\overrightarrow{Q_1Q_2Q_3}$ that is equal to the triangle $\overrightarrow{A_1A_2A_3}$ and has the same orientation (Fig. 22).

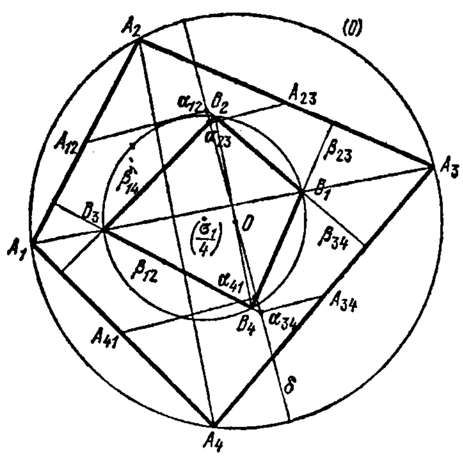

Fig. 23

Problem 19. Let $A_1A_2A_3A_4$ be a convex quadrangle inscribed in a circle (O). Denote by $A_{12}, A_{23}, A_{34}, A_{41}$ the midpoints of its sides A_1A_2, A_2A_3, A_3A_4, A_4A_1. Let δ be some diameter of (O). Denote by α_{12}, α_{23}, α_{34}, α_{41} the orthogonal projections of the points $A_{12}, A_{23}, A_{34}, A_{41}$ on the diameter δ. Draw through the points $\alpha_{12}, \alpha_{23}, \alpha_{34}, \alpha_{41}$ straight lines β_{34}, β_{14}, β_{12}, β_{23} that are respectively perpendicular to the straight lines A_3A_4, A_4A_1, A_1A_2, A_2A_3. Prove that the lines $\beta_{34}, \beta_{14}, \beta_{12}, \beta_{23}$ form a quadrangle $\overrightarrow{B_1B_2B_3B_4}$ similar to the given one but with orientation opposite that of the quadrangle $\overrightarrow{A_1A_2A_3A_4}$ (B_1 is the point of intersection of the lines β_{23} and β_{34}, B_2 is the point of intersection of the lines β_{14} and β_{34}, B_3 is the point of intersection of the lines β_{12} and β_{14}, B_4 is the point of intersection of the lines β_{12} and β_{23}; Fig. 23).

Solution. Regard (O) as the unit circle, and take the x-axis so that the diameter δ lies on it. Let z_1, z_2, z_3, z_4 be the respective affixes of the points A_1, A_2, A_3, A_4. The midpoint A_{12} of segment A_1A_2 has the affix

$$a_{12} = \frac{z_1 + z_2}{2}$$

and its projection α_{12} on the line δ (the x-axis) has the affix

$$\tau_{12} = \frac{a_{12} + \bar{a}_{12}}{2} = \frac{z_1 + z_2 + \dfrac{1}{z_1} + \dfrac{1}{z_2}}{4} = \frac{(1 + z_1 z_2)(z_1 + z_2)}{4z_1 z_2}.$$

The equation of the perpendicular β_{34} dropped from the point α_{12} to the line A_3A_4 (the slope of which is equal to $-z_3z_4$) is of the form

$$z - \tau_{12} = z_3 z_4(\bar{z} - \bar{\tau}_{12})$$

or

$$z - \frac{(1 + z_1 z_2)(z_1 + z_2)}{4z_1 z_2} = z_3 z_4 \left[\bar{z} - \frac{(1 + z_1 z_2)(z_1 + z_2)}{4z_1 z_2} \right]$$

or

$$z - z_3 z_4 \bar{z} = \frac{(z_1 + z_2)(1 + z_1 z_2)(1 - z_3 z_4)}{4z_1 z_2}. \qquad (52)$$

Similarly, the equation of the perpendicular β_{23} dropped from the point α_{14} to the line A_2A_3 is of the form

$$z - z_2 z_3 \bar{z} = \frac{(z_1 + z_4)(1 + z_1 z_4)(1 - z_2 z_3)}{4z_1 z_4}. \qquad (53)$$

Multiplying both sides of equation (52) by z_2, both sides of (53) by $-z_4$, and adding termwise, we obtain the affix $z = b_1$ of point B_1, the point of intersection of the lines β_{34} and β_{23}:

$$(z_2 - z_4)\, b_1 = \frac{(z_1 + z_2)(1 + z_1 z_2)(1 - z_3 z_4) - (z_1 + z_4)(1 + z_1 z_4)(1 - z_2 z_3)}{4z_1}$$

$$= \frac{1}{4z_1}(z_1 + z_2 + z_1^2 z_2 + z_2^2 z_1 - z_1 z_3 z_4 - z_2 z_3 z_4 - z_1^2 z_2 z_3 z_4$$

$$- z_2^2 z_1 z_3 z_4 - z_1 - z_4 - z_1 z_4^2 - z_4 z_1^2 + z_1 z_2 z_3 + z_2 z_3 z_4 + z_4^2 z_1 z_2 z_3 + z_1^2 z_2 z_3 z_4)$$

$$= \frac{1}{4z_1}[z_2 - z_4 + z_1^2(z_2 - z_4) + z_1 z_3(z_2 - z_4)$$

$$+ z_1(z_2^2 - z_4^2) - z_1 z_2 z_3 z_4(z_2 - z_4)],$$

whence

$$b_1 = \frac{1 + z_1^2 + z_1 z_3 + z_1 z_2 + z_1 z_4 - z_1 z_2 z_3 z_4}{4z_1} = \frac{1 - \sigma_4}{4z_1} + \frac{\sigma_1}{4},$$

where

$$\sigma_1 = z_1 + z_2 + z_3 + z_4, \quad \sigma_4 = z_1 z_2 z_3 z_4.$$

Similarly we find the affixes b_2, b_3, b_4 of points B_2, B_3, B_4:

$$b_2 = \frac{1 - \sigma_4}{4z_2} + \frac{\sigma_1}{4},$$

$$b_3 = \frac{1 - \sigma_4}{4z_3} + \frac{\sigma_1}{4},$$

$$b_4 = \frac{1 - \sigma_4}{4z_4} + \frac{\sigma_1}{4}.$$

To summarize:

$$b_k = \frac{\sigma_1}{4} + \frac{1 - \sigma_4}{4}\,\bar{z}_k \quad (k = 1, 2, 3, 4).$$

From this relation it follows that the points B_1, B_2, B_3, B_4 lie on the circle $(B_1B_2B_3B_4)$, the affix of whose centre is equal to $\sigma_1/4$ and the radius is $|1 - \sigma_4|/4$. Since $|\sigma_4| = 1$, it follows that the radius can vary from 0 to 1/2 (σ_4 can assume the values $\pm$ 1).

Thus, if $\sigma_4 = 1$, that is the Boutain point of the quadrangle is the unit point, then the quadrangle $B_1B_2B_3B_4$ contracts to a point: all the lines $\beta_{34}, \beta_{14}, \beta_{12}, \beta_{23}$ pass through the centroid $G\,(\sigma_1/4)$ of the system of points A_1, A_2, A_3, A_4, to which are assigned equal masses.

If $\sigma_1 = 0$, that is, if the centroid of four points A_1, A_2, A_3, A_4 coincides with the centre O of the circle $(A_1A_2A_3A_4) = (O)$, then $(A_1A_2A_3A_4)$ and $(B_1B_2B_3B_4)$ are concentric circles.

Finally, if the unit point does not coincide with any one of the four Boutain points of the quadrangle $A_1A_2A_3A_4$ (these Boutain points form the vertices of a square inscribed in the circle (O)), then the quadrangle $\overrightarrow{B_1B_2B_3B_4}$ does not degenerate. It is an image of the quadrangle $A_1A_2A_3A_4$ under a similarity transformation of the second kind:

$$u = \frac{1 - \sigma_4}{4}\,\bar{z} + \frac{\sigma_1}{4}.$$

We first take the points A_1', A_2', A_3', A_4' that are symmetric respectively to the points A_1, A_2, A_3, A_4 with respect to the x-axis, and then the quadrangle $\overrightarrow{A_1'A_2'A_3'A_4'}$ is rotated about point O through the angle $\arg\dfrac{1 - \sigma_4}{4}$ and the rotated quadrangle $\overrightarrow{A_1''\,A_2''\,A_3''\,A_4''}$ is subjected to a homothetic transformation with centre O and ratio $|1 - \sigma_4|/4$; we get a quadrangle $\overrightarrow{A_1'''A_2'''A_3'''A_4'''}$; finally, the last quadrangle is subjected to a parallel translation determined by the directed line segment $\overrightarrow{OG}$, where the affix g of point G is equal to $\sigma_1/4$ (G is the centroid of the system of four points A_1, A_2, A_3, A_4 to which are assigned equal masses). We obtain a quadrangle $\overrightarrow{B_1B_2B_3B_4}$.

Of all these transformations, only the first one (symmetry about the x-axis) changes the orientation and, hence, $\overrightarrow{A_1A_2A_3A_4} \downarrow\uparrow \overrightarrow{B_1B_2B_3B_4}$, and the quadrangles $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are similar.

Problem 20. A triangle ABC is inscribed in a circle with centre O; A_0, B_0, C_0 are the centres of the circles $(OBC), (OCA), (OAB)$; A_1, B_1, C_1 are points symmetric to the points A_0, B_0, C_0 about BC, CA, AB respectively. Prove that the orthocentre H of $\triangle ABC$ is the centre of one of the circles (S) tangent to the straight lines B_1C_1, C_1A_1, A_1B_1 and that the

circle (S) passes through the centre of the Euler circle of $\triangle ABC$. Prove that the radius of (S) is equal to $\frac{1}{2}OH$.

Proof. Assume (ABC) to be the unit circle. Draw tangent lines to (ABC) at the points A, B, C. They form a triangle $A_2B_2C_2$. The circle $(ABC) = (O)$ is inscribed in $\triangle A_2B_2C_2$ if ABC is an acute-angled triangle, and escribed if ABC is an obtuse-angled triangle; if, for example, the angle C is obtuse, then the circle (O) is escribed in the angle C_2 of triangle $A_2B_2C_2$.

In the quadrangle OBA_2C the angles B and C are equal to $\pi/2$, and so a circle can be circumscribed about it; the segment OA_2 becomes a diameter of that circle and, hence, its centre is the midpoint of segment OA_2. But the circle (OBC) coincides of course with the circle (OBA_2C); hence, the centre of (OBC) is the midpoint A_0 of segment OA_2.

Let z_1, z_2, z_3 be the affixes of the points A, B, C. Then the affix a_3 of the midpoint A_3 of segment BC is

$$a_3 = \frac{z_2 + z_3}{2},$$

and since point A_2 is obtained from A_3 by inversion with the inversion circle (ABC), it follows that the affix a_2 of point A_2 is

$$a_2 = \frac{1}{\bar{a}_3} = \frac{2}{\bar{z}_2 + \bar{z}_3}.$$

The affix a_0 of point A_0 will therefore be

$$a_0 = \frac{1}{\bar{z}_2 + \bar{z}_3}.$$

The affix a_1 of point A_1, symmetric to point A_0 about the line BC, is found from the relation

$$\frac{a_0 + a_1}{2} = a_3,$$

whence

$$a_1 = 2a_3 - a_0$$

$$= z_2 + z_3 - \frac{1}{\bar{z}_2 + \bar{z}_3} = z_2 + z_3 - \frac{z_2 z_3}{z_2 + z_3} = \sigma_1 - \frac{z_2 z_3}{z_2 + z_3} - z_1 = \sigma_1 - \frac{\sigma_2}{z_2 + z_3}.$$

The affixes b_1 and c_1 of points B_1 and C_1 are similar in form. Thus,

$$a_1 = \sigma_1 - \frac{\sigma_2}{z_2 + z_3} = \sigma_1 - \frac{1}{2}\sigma_2 \bar{a}_2,$$

$$b_1 = \sigma_1 - \frac{\sigma_2}{z_3 + z_1} = \sigma_1 - \frac{1}{2}\sigma_2 \bar{b}_2,$$

$$c_1 = \sigma_1 - \frac{\sigma_2}{z_1 + z_2} = \sigma_1 - \frac{1}{2}\sigma_2 \bar{c}_2.$$

From this it follows that $\triangle A_1B_1C_1$ is obtained from $\triangle A_2B_2C_2$ via the similarity transformation

$$u = \sigma_1 - \frac{1}{2}\sigma_2 \bar{z}.$$

This is a similarity transformation of the second kind: first symmetry is executed about the x-axis ($z \to \bar{z}$), then a rotation about the point O through the angle $\arg(-\sigma_2)$, and then a homothetic transformation with centre O and ratio $|\sigma_2|/2 = |\sigma_1|/2$; finally, a translation defined by the directed line segment $\overrightarrow{OH}$ (since σ_1 is the affix of point H).

As a result of these transformations, $\triangle \overrightarrow{A_2B_2C_2}$ goes into $\triangle \overrightarrow{A_1B_1C_1}$, which is similar to $\triangle \overrightarrow{A_2B_2C_2}$ but has opposite orientation.

Furthermore, since under the transformation

$$u = \sigma_1 - \frac{1}{2}\sigma_2 \bar{z}$$

the centre O of the circle (ABC) that is tangent to the sides of $\triangle A_2B_2C_2$ goes into the orthocentre H of $\triangle ABC$, it follows that the circle (S), into which the circle (O) passes, will touch the sides of $\triangle A_1B_1C_1$ (into which $\triangle A_2B_2C_2$ passes), and point H will be the centre of (S).

Under the succession of transformations resulting in the transformation

$$u = \sigma_1 - \frac{1}{2}\sigma_2 \bar{z},$$

the radius of (ABC) changes only under the homothetic transformation $(O, |\sigma_1|/2)$; consequently, the radius of (S) is equal to

$$\frac{|\sigma_1|}{2} = \frac{1}{2} OH$$

since the radius R of (ABC) is equal to 1.

Problem 21. Let $A_1A_2A_3A_4$ be an arbitrary quadrangle inscribed in a circle (O) and let P be an arbitrary point. Denote by $P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}$ the points symmetric to point P about the straight lines A_1A_2, A_1A_3, A_1A_4, A_2A_3, A_2A_4, A_3A_4. Let R, S, T be the respective midpoints of segments $P_{12}, P_{34}, P_{13}, P_{24}, P_{14}, P_{23}$, and let O' be a point symmetric to the centre O of (O) about the centroid of the system of four points A_1, A_2, A_3, A_4 (to which are assigned equal masses; Fig. 24).

Prove that the points R, S, T, O' lie on one straight line.

Solution. Assume (O) to be the unit circle. Let z_1, z_2, z_3, z_4 be the respective affixes of the points A_1, A_2, A_3, A_4. Denote by $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ the basic symmetric polynomials of the (complex) numbers z_i, $i = 1, 2, 3, 4$:

$$\sigma_1 = z_1 + z_2 + z_3 + z_4,$$

$$\sigma_2 = z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4,$$

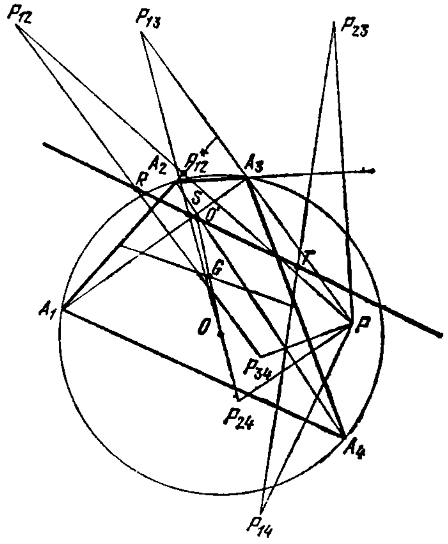

Fig. 24

$$\sigma_3 = z_2 z_3 z_4 + z_1 z_3 z_4 + z_1 z_2 z_4 + z_1 z_2 z_3,$$

$$\sigma_4 = z_1 z_2 z_3 z_4.$$

The affix g of the centroid G of the system of points A_1, A_2, A_3, A_4 is equal to $\sigma_1/4$, hence, the affix o' of point O', which is symmetric to point O about point G, is equal to

$$o' = \sigma_1/2.$$

The equation of the straight line A_1A_2 is

$$z + z_1 z_2 \bar{z} = z_1 + z_2. \tag{54}$$

The equation of the perpendicular dropped from point P to the line A_1A_2 is of the form

$$z - p = z_1 z_2(\bar{z} - \bar{p})$$

or

$$z - z_1 z_2 \bar{z} = p - z_1 z_2 \bar{p}, \tag{55}$$

where p is the affix of point P. Adding termwise equations (54) and (55), we find the affix $z = p_{12}^*$ of the projection P_{12}^* of point P on the line A_1A_2:

$$p_{12}^* = \frac{1}{2}(z_1 + z_2 + p - z_1 z_2 \bar{p}).$$

The affix p_{12} of point P_{12}, which is symmetric to point P about line A_1A_2, is found from the relation

$$\frac{p+p_{12}}{2}=p_{12}^*,$$

whence

$$p_{12}=2p_{12}^*-p=z_1+z_2-z_1z_2\bar{p}. \tag{56}$$

Of similar form is the affix p_{34} of point P_{34}, which is symmetric to point P about line A_3A_4:

$$p_{34}=z_3+z_4-z_3z_4\bar{p}. \tag{57}$$

From (56) and (57) we find the affix r of the midpoint R of segment $P_{12}P_{34}$:

$$r=\frac{1}{2}[\sigma_1-(z_1z_2+z_3z_4)\bar{p}]. \tag{58}$$

The slope of the line $O'R$ is

$$\varkappa=\frac{\dfrac{\sigma_1}{2}-r}{\dfrac{\bar{\sigma}_1}{2}-\bar{r}}=\frac{\dfrac{\sigma_1}{2}-\dfrac{1}{2}[\sigma_1-(z_1z_2+z_3z_4)\bar{p}]}{\dfrac{\bar{\sigma}_1}{2}-\dfrac{1}{2}\left[\bar{\sigma}_1-\left(\dfrac{1}{z_1z_2}+\dfrac{1}{z_3z_4}\right)p\right]}=\frac{\bar{p}}{p}\sigma_4,$$

and therefore the equation of $O'R$ is

$$z-\frac{\sigma_1}{2}=\frac{\bar{p}}{p}\sigma_4\left(\bar{z}-\frac{\sigma_3}{2\sigma_4}\right).$$

This equation may be rewritten thus:

$$2pz-2\bar{p}\,\sigma_4\bar{z}+\sigma_3\bar{p}-p\sigma_1=0. \tag{59}$$

Since this equation involves only symmetric polynomials of the affixes z_1, z_2, z_3, z_4 of points A_1, A_2, A_3, A_4, the equations of the straight lines $O'S$ and $O'T$ will be the same as equation (59), that is, the points O', S, T, R lie on one straight line. Incidentally, we can see directly that the affixes s and t of points S and T,

$$s=\frac{1}{2}[\sigma_1-(z_1z_3+z_2z_4)\bar{p}]$$

$$t=\frac{1}{2}[\sigma_1-(z_1z_4+z_2z_3)\bar{p}]$$

satisfy the equation (59).

Remark. Since equation (59) passes into an equivalent equation if p is replaced by λp, where λ is an arbitrary real number, it follows that

the straight line (59) does not change if point P describes a straight line passing through the centre O of circle (O) (provided, of course, the point O itself is excluded from the line).

Problem 22. Let A_1, B_1, C_1 be projections of the vertices A, B, C of $\triangle ABC$ on the diameter δ of the circle (ABC). Denote by A_2, B_2, C_2 the points symmetric to points A_1, B_1, C_1 respectively with respect to the mid-perpendiculars of the sides BC, CA, AB of $\triangle ABC$. Let A_3, B_3, C_3 be the midpoints of segments BC, CA, AB, and let A_4, B_4, C_4 be the midpoints of segments A_2A_3, B_2B_3, C_2C_3. Prove that $\triangle \overrightarrow{ABC}$ and $\triangle \overrightarrow{A_4B_4C_4}$ are similar and have opposite orientations. Prove that if the diameter δ of $(ABC) = (O)$ rotates about the point O, then the centre S of the circle $(S) = (A_4B_4C_4)$ describes a circle (Ω) that is concentric with (ABC), and the rotations are in opposite directions. The radius of (S) is equal to $\frac{1}{4}$ OH, where H is the orthocentre of $\triangle ABC$. The radius of (Ω) is equal to $R/4$, where R is the radius of (ABC) (Fig. 25).

Solution. Take $(O) = (ABC)$ to be the unit circle, and let the diameter δ be the x-axis. Let z_1, z_2, z_3 be the affixes of the points A, B, C. The affix a_1 of the projection A_1 of point A on the straight line δ is

$$a_1 = \frac{z_1 + \bar{z}_1}{2}.$$

Since line BC has a slope of $-z_2z_3$, it follows that the equation of the straight line OA_3 is of the form

$$z = z_2 z_3 \bar{z}$$

and the equation of the perpendicular dropped from point A_1 to line OA_3 is

$$z - \frac{z_1 + \bar{z}_1}{2} = - z_2 z_3 \left(\bar{z} - \frac{z_1 + \bar{z}_1}{2}\right).$$

From the system of equations

$$\left.\begin{aligned} & z - z_2 z_3 \bar{z} = 0, \\ & z + z_2 z_3 \bar{z} = \frac{z_1 + \bar{z}_1}{2}(1 + z_2 z_3), \end{aligned}\right\} \qquad (60)$$

we find the affix $z = a_2^*$ of the projection A_2^* of point A_1 on the midperpendicular OA_3 of segment BC:

$$a_2^* = \frac{z_1 + \bar{z}_1}{4}(1 + z_2 z_3).$$

The affix a_2 of point A_2 is found from

$$\frac{a_1 + a_2}{2} = a_2^*,$$

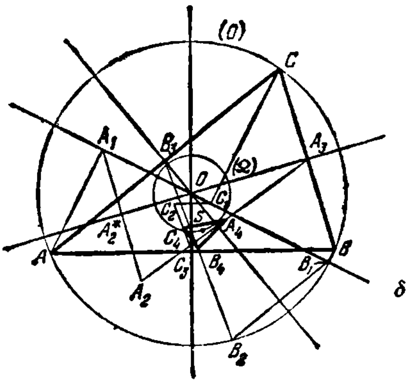

Fig. 25

whence

$$a_2 = 2a_2^* - a_1 = \frac{z_1 + \bar{z}_1}{2}(1 + z_2 z_3) - \frac{z_1 + \bar{z}_1}{2} = \frac{z_1 + \bar{z}_1}{2} z_2 z_3.$$

The affix a_4 of midpoint A_4 of segment A_2A_3 is

$$a_4 = \frac{a_2 + a_3}{2},$$

where a_3 is the affix of the midpoint A_3 of segment BC, that is,

$$a_3 = \frac{z_2 + z_3}{2}.$$

To summarize:

$$a_4 = \frac{1}{2}\left(\frac{z_1 + \bar{z}_1}{2} z_2 z_3 + \frac{z_2 + z_3}{2}\right) = \frac{1}{4}\left(\sigma_3 + \frac{z_2 z_3}{z_1} + z_2 + z_3\right) = \frac{1}{4}\left(\sigma_3 + \frac{\sigma_2}{z_1}\right).$$

Of similar form are the affixes b_4 and c_4 of points B_4 and C_4. We have

$$\left.\begin{aligned} a_4 &= \frac{\sigma_3}{4} + \frac{\sigma_2}{4z_1}, \\ b_4 &= \frac{\sigma_3}{4} + \frac{\sigma_2}{4z_2}, \\ c_4 &= \frac{\sigma_3}{4} + \frac{\sigma_2}{4z_3}. \end{aligned}\right\} \tag{61}$$

From this it follows that the points A_4, B_4, C_4 are obtained from the points A, B, C via the similarity transformation

$$u = \frac{\sigma_3}{4} + \frac{\sigma_2}{4}\bar{z}$$

of the second kind, and, hence, $\triangle \overrightarrow{ABC}$ and $\triangle \overrightarrow{A_4B_4C_4}$ have opposite orientations (and are similar).

From the relations (61) it also follows that the points A_4, B_4, C_4 lie on the circle (S) with centre S whose affix s is

$$s = \sigma_3/4$$

and the radius of the circle (S) is equal to

$$\rho = \frac{|\sigma_2|}{4} = \frac{|\sigma_1|}{4} = \frac{1}{4} OH.$$

The distance d from the centre S of $(A_4B_4C_4)$ to the centre O of $(O) = (ABC)$ is

$$d = \frac{|\sigma_3|}{4} = \frac{1}{4} = \frac{R}{4} \quad (R = 1).$$

This means that when the diameter δ is rotated about the point O, the centre S of $(A_4B_4C_4)$ describes a circle (Ω) whose radius is equal to $\frac{1}{4} = R/4$, where R is the radius of (ABC). If the diameter δ is turned through an angle α, then the affix of its end, which had an affix equal to 1, will have an affix $\beta = \cos\alpha + i\sin\alpha$; and if this point with affix β is taken as the new unit point, then the new affixes of the points A, B, C will be $\frac{z_1}{\beta}, \frac{z_2}{\beta}, \frac{z_3}{\beta}$, and the new affix of the new centre S^* of the new triangle $A_4^*B_4^*C_4^*$ will be σ_3/β^3. This means the affix of point S^* in the initial system will be σ_3/β^2. From this it follows that the radius OS of the circle (Ω) will rotate in a direction opposite that of the diameter δ, and the angular velocity of rotation of OS will be twice the angular velocity of rotation of δ.

Remark. Let us also find the ratio

$$\frac{(A_4B_4C_4)}{(ABC)}.$$

We have

$$(A_4B_4C_4) = \frac{i}{4}\begin{vmatrix} \frac{\sigma_3}{4} + \frac{\sigma_2}{4z_1} & \frac{\bar\sigma_3}{4} + \frac{\bar\sigma_2}{4} z_1 & 1 \\ \frac{\sigma_3}{4} + \frac{\sigma_2}{4z_2} & \frac{\bar\sigma_3}{4} + \frac{\bar\sigma_2}{4} z_2 & 1 \\ \frac{\sigma_3}{4} + \frac{\sigma_2}{4z_3} & \frac{\bar\sigma_3}{4} + \frac{\bar\sigma_2}{4} z_3 & 1 \end{vmatrix} = \frac{i}{4}\begin{vmatrix} \frac{\sigma_2}{4z_1} & \frac{\bar\sigma_2}{4} z_1 & 1 \\ \frac{\sigma_2}{4z_2} & \frac{\bar\sigma_2}{4} z_2 & 1 \\ \frac{\sigma_2}{4z_3} & \frac{\bar\sigma_2}{4} z_3 & 1 \end{vmatrix}$$

$$= \frac{i}{4}\,\frac{\sigma_2\bar\sigma_2}{16}\begin{vmatrix} \bar z_1 & z_1 & 1 \\ \bar z_2 & z_2 & 1 \\ \bar z_3 & z_3 & 1 \end{vmatrix} = -\frac{|\sigma_2|^2}{16}(ABC) = -\frac{|\sigma_1|^2}{16}(ABC) = -\frac{OH^2}{16}(ABC) \tag{62}$$

whence

$$\frac{(A_4B_4C_4)}{(ABC)} = -\frac{OH^2}{16}.$$

From this it also follows that $\overrightarrow{A_4B_4C_4} \downarrow\uparrow \overrightarrow{ABC}$.

Problem 23. Given a triangle T such that there exists a straight line τ intersecting its sides at angles equal to the angles of the triangle, that is *,

$$\begin{aligned}(AB, \tau) &= (AC, AB) = A,\\ (BC, \tau) &= (BA, BC) = B, \qquad (63)\\ (CA, \tau) &= (CB, CA) = C.\end{aligned}$$

1°. Find the angles of the triangle T.

2°. Through the points A, B, C draw straight lines parallel to the line τ. Denote by A_1, C_1, B_1, respectively, the points of intersection of these lines with the straight lines BC, CA, AB. Prove that $\triangle\overrightarrow{ACB}$ and $\triangle\overrightarrow{B_1C_1A_1}$, are similar where

$$A \leftrightarrow B_1, \; C \leftrightarrow C_1, \; B \leftrightarrow A_1,$$

and that they have opposite orientations.

3°. A transversal τ_1 is constructed for $\triangle B_1C_1A_1$ in the same way as the transversal τ was constructed for $\triangle ABC$. Let the straight lines passing through the points A_1, B_1, C_1 respectively and parallel to the straight line τ_1 intersect the lines B_1C_1, C_1A_1, A_1B_1 respectively, in the points A_2, B_2, C_2.

Prove that $\triangle\overrightarrow{ABC}$ and $\triangle\overrightarrow{C_2B_2A_2}$ are homothetic and have the same orientation, that is, prove that the straight lines AC_2, BB_2, CA_2 pass through the same point S and $AB \| C_2B_2$, $BC \| A_2B_2$, $CA \| A_2C_2$.

4°. Prove that the diameter of the circle (ABC) passing through the point S is perpendicular to the straight line τ (Fig. 26).

Solution. 1°. Suppose that $C < B < A$. By the Chasles theorem we find

$$B = (BA, BC) = (BA, \tau) + (\tau, BC) = A - B,$$

whence

$$A = 2B.$$

Furthermore,

$$C = (CB, CA) = (CB, \tau) + (\tau, CA) = B - C$$

---

* The symbol (p, q) is used to denote the *oriented* angle from straight line p to straight line q (p and q lie on an oriented plane). If α is one of the values of the angle (p, q), then all values of this angle are given by the formula

$$(p, q) = \alpha + k\pi,$$

where k assumes all integral values; stated differently,

$$(p, q) \equiv \alpha \pmod{\pi}.$$

The *Chasles theorem* for three straight lines p, q, r lying on an oriented plane holds:

$$(p, q) + (q, r) \equiv (p, r) \pmod{\pi}.$$

The relation (63) is to be understood as follows: one of the values (CA, τ) is equal to C, one of the values (AB, BC) is equal to B, and so on.

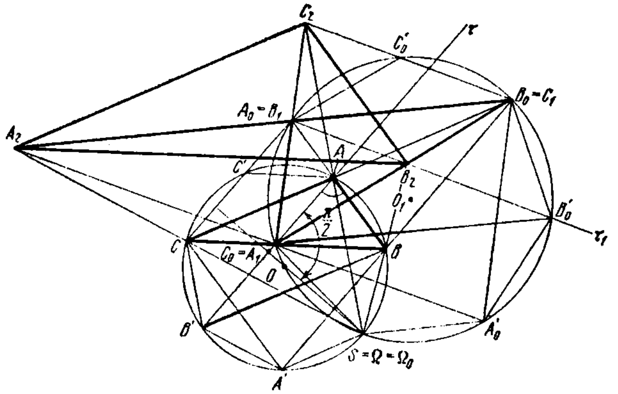

Fig. 26

whence

$$B = 2C.$$

Thus

$$A = 2B = 4C$$

and so

$$C = \frac{\pi}{7}, \quad B = \frac{2\pi}{7}, \quad A = \frac{4\pi}{7}.$$

2°. Assume (ABC) to be the unit circle, and A the unit point. The vertices A, B, C of $\triangle\, ABC$ will be the vertices of a regular heptagon $AC'CB'A'\Omega B$ inscribed in the circle (ABC). Setting $\alpha = \cos\ (2\pi/7) + i \sin (2\pi/7)$ we find the affixes a, c', c, b', a', ω, b of the vertices of this heptagon:

$$a = 1,\ c' = \alpha\ (= \alpha^1),\ c = \alpha^2,\ b' = \alpha^3,\ a' = \alpha^4,\ \omega = \alpha^5,\ b = \alpha^6.$$

The quadrangle $AC'CA_1$ is a rhombus because we can take the straight line AB' for the straight line τ (then all of the relations (63) of the statement of the problem will be fulfilled) and, hence, by construction we have $\tau = AA_1 \| CC'$, $CB \| AC'$, $C'C = C'A$. The quadrangle $CA'BB_1$ is also a rhombus since its opposite sides are parallel: $A'B \| \tau \|\ CB_1$, $ABB_1 \| A'C$ and $A'C = A'B$. Similarly, we see that the quadrangle $B'BC_1A$ is a rhombus as well.

From this we can find the affixes of the points $A_1B_1C_1$; since

$$\overrightarrow{A_1C} + \overrightarrow{A_1A} = \overrightarrow{A_1C'},$$

it follows that

$$c - a_1 + a - a_1 = c' - a_1,$$

whence

$$a_1 = c + a - c' = \alpha^2 + 1 - \alpha.$$

Furthermore, from the relation

$$\overrightarrow{A'C} + \overrightarrow{A'B} = \overrightarrow{A'B_1}$$

we find

$$c - a' + b - a' = b_1 - a'$$

and, consequently,

$$b_1 = c + b - a' = \alpha^2 + \alpha^6 - \alpha^4.$$

Finally, from the relation

$$\overrightarrow{B'A} + \overrightarrow{B'B} = \overrightarrow{B'C_1}$$

we find

$$a - b' + b - b' = c_1 - b',$$

$$c_1 = a + b - b' = 1 + \alpha^5 - \alpha^3.$$

To summarize:

$$a_1 = 1 - \alpha + \alpha^2,$$

$$b_1 = \alpha^2 - \alpha^4 + \alpha^6,$$

$$c_1 = 1 - \alpha^3 + \alpha^6.$$

The triangles $\overrightarrow{ACB}$ and $\overrightarrow{B_1C_1A_1}$ are similar and have opposite orientations if and only if the determinant

$$\Delta = \begin{vmatrix} \bar{a} & b_1 & 1 \\ \bar{c} & c_1 & 1 \\ \bar{b} & a_1 & 1 \end{vmatrix}$$

is equal to zero. We have

$$\Delta = \begin{vmatrix} 1 & \alpha^2 - \alpha^4 + \alpha^6 & 1 \\ \bar{\alpha}^2 & 1 - \alpha^3 + \alpha^6 & 1 \\ \bar{\alpha}^6 & 1 - \alpha + \alpha^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & \alpha^2 - \alpha^4 + \alpha^6 & 1 \\ \alpha^5 & 1 - \alpha^3 + \alpha^6 & 1 \\ \alpha & 1 - \alpha + \alpha^2 & 1 \end{vmatrix}$$

$$= 1 - \alpha^3 + \alpha^6 + \alpha^5 - \alpha^6 + \alpha^7 + \alpha^3 - \alpha^5 + \alpha^7 - \alpha + \alpha^4 - \alpha^7 - 1$$
$$+ \alpha - \alpha^2 - \alpha^7 + \alpha^9 - \alpha^{11} \equiv 0.$$

Now we find the centre of similarity* of $\triangle\overrightarrow{ACB}$ and $\triangle\overrightarrow{B_1C_1A_1}$. Let us consider the similarity transformation of the second kind under which

---

* The *centre of similarity* of two mirror-similar triangles (that is similar triangles with opposite orientations) is understood to be the fixed point of a similarity transformation of the second kind that carries one of these triangles into the other.

the points A_1 and B_1 go into points B and A (then point C_1 goes into C). Let z be the affix of an arbitrary point M of the plane, and let $M'(u)$ be its image under the indicated similarity transformation. Then

$$\begin{vmatrix} z & u & 1 \\ b_1 & 1 & 1 \\ a_1 & \alpha & 1 \end{vmatrix} = 0$$

or

$$u(a_1 - b_1) + z(1 - \alpha) + \alpha b_1 - a_1 = 0$$

or

$$u(1 - \alpha + \alpha^4 - \alpha^6) + z(1 - \alpha) + \alpha^3 - \alpha^5 + \alpha^7 - 1 + \alpha - \alpha^2 = 0,$$

whence, cancelling $1 - \alpha$, we have

$$u(\alpha^5 + \alpha^4 + 1) + z + \alpha(\alpha^3 + \alpha^2 + 1) = 0.$$

The fixed point of the similarity transformation satisfies the condition

$$\bar{z}(\alpha^5 + \alpha^4 + 1) + z + \alpha^4 + \alpha^3 + \alpha = 0, \tag{64}$$

whence (passing over to conjugate numbers) we have

$$z(\bar{\alpha}^5 + \bar{\alpha}^4 + 1) + \bar{z} + \bar{\alpha}^4 + \bar{\alpha}^3 + \bar{\alpha} = 0$$

or, multiplying the left-hand side by $\alpha^7 = 1$,

$$z(\alpha^2 + \alpha^3 + 1) + \bar{z} + \alpha^3 + \alpha^4 + \alpha^6 = 0.$$

From the last relation we find

$$\bar{z} = -z(\alpha^2 + \alpha^3 + 1) - \alpha^3 - \alpha^4 - \alpha^6$$

and equation (64) takes the form

$$-z(\alpha^2 + \alpha^3 + 1)(\alpha^5 + \alpha^4 + 1) - (\alpha^5 + \alpha^4 + 1)(\alpha^3 + \alpha^4 + \alpha^6) + z + \alpha^3 + \alpha^4 + \alpha = 0$$

or

$$z(1 - \alpha^7 - \alpha^6 - \alpha^2 - \alpha^8 - \alpha^7 - \alpha^3 - \alpha^5 - \alpha^4 - 1) = \alpha^8 + \alpha^9 + \alpha^{11} + \alpha^7 + \alpha^8 + \alpha^{10} + \alpha^3 + \alpha^4 + \alpha^6 - \alpha^3 - \alpha^4 - \alpha$$

or

$$z(-\alpha^6 - \alpha^5 - \alpha^4 - \alpha^3 - \alpha^2 - \alpha - 2) = \alpha^6 + \alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1.$$

But $\alpha^7 = 1$, and since $\alpha - 1 \neq 0$, it follows that

$$\alpha^6 + \alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$$

and the last relation takes the form

$$-z = -\alpha^5,$$

whence

$$z = \alpha^5.$$

That is, the fixed point of the similarity transformation of the second kind that carries $\triangle \overrightarrow{B_1C_1A_1}$ into $\triangle \overrightarrow{ABC}$ is the point Ω.

Setting

$$\alpha^5 + \alpha^4 + 1 = \lambda, \ \alpha^3 + \alpha^2 + 1 = \mu,$$

we find the relation

$$\lambda \bar{u} + z + \alpha\mu = 0 \tag{65}$$

that relates the affix z of the arbitrary point M of the plane with the affix u of its image M' under the similarity transformation at hand. The points B_1 and A_1 go into points A and B under this similarity transformation, so that

$$\bar{a} = -\frac{b_1 + \alpha\mu}{\lambda}, \ \bar{b} = -\frac{a_1 + \alpha\mu}{\lambda},$$

whence

$$\bar{a} - \bar{b} = \frac{a_1 - b_1}{\lambda}$$

and, thus,

$$\frac{A_1B_1}{AB} = |\lambda|.$$

However,

$$\alpha^5 + \alpha^4 + 1 = \alpha^{-2} + \alpha^{-3} + 1,$$

$$\overline{\alpha^5 + \alpha^4 + 1} = \alpha^2 + \alpha^3 + 1;$$

consequently,

$$\begin{aligned}|\lambda|^2 &= (\alpha^5 + \alpha^4 + 1)(\alpha^2 + \alpha^3 + 1) \\ &= \alpha^7 + \alpha^8 + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^4 + \alpha^2 + \alpha^3 + 1 \\ &= 2 + \alpha^6 + \alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 2,\end{aligned}$$

whence

$$\frac{A_1B_1}{AB} = \sqrt{2}, \ \frac{AB}{A_1B_1} = \frac{1}{\sqrt{2}}$$

is the proportionality factor that carries $\triangle \overrightarrow{B_1C_1A_1}$ into $\triangle \overrightarrow{ACB}$; it is, equal to $1/\sqrt{2}$.

3°. Let O_1 be the centre of the circle $(A_1B_1C_1)$. The point O_1 is the image of point O under the indicated similarity transformation (item 2°). Therefore, the affix o_1 of point O_1 is found from equation (65) by setting $u = 0$ in the equation:

$$o_1 = -\alpha(\alpha^3 + \alpha^2 + 1).$$

Let us consider a regular heptagon inscribed in a circle $(A_1B_1C_1)$ with the vertices arranged in the following order:

$$A_0C_0\Omega_0A_0'B_0'B_0C_0',$$

where $A_0 = B_1$, $B_0 = C_1$, $C_0 = A_1$. Such a regular heptagon exists since the angles of $\triangle B_1A_1C_1$ are

$$B_1 = \frac{4\pi}{7}, \quad A_1 = \frac{2\pi}{7}, \quad C_1 = \frac{\pi}{7}$$

and, hence, A_1B_1 is one of the sides of the heptagon and C_1 is its third vertex. The affixes of the vertices of the heptagon are

$$o_1 + (a_0 - o_1)\,\alpha^{k-1} \quad (k = 1, 2, 3, 4, 5, 6, 7).$$

Expanded, they are:

$$a_0 = b_1 = \alpha^2 - \alpha^4 + \alpha^6,$$

$$c_0 = a_1 = \alpha^2 - \alpha + 1,$$

$$\omega_0 = -\alpha^4 - \alpha^3 - \alpha + (\alpha^2 + \alpha^6 + \alpha^3 + \alpha)\,\alpha^2$$
$$= -\alpha^4 - \alpha^3 - \alpha + \alpha^4 + \alpha^8 + \alpha^5 + \alpha^3 = \alpha^5,$$

$$a_0' = -\alpha^4 - \alpha^3 - \alpha + (\alpha^2 + \alpha^6 + \alpha^3 + \alpha)\,\alpha^3$$
$$= -\alpha^4 - \alpha^3 - \alpha + \alpha^5 + \alpha^9 + \alpha^6 + \alpha^4 = \alpha^6 + \alpha^5 - \alpha^3 + \alpha^2 - \alpha,$$

$$b_0' = -\alpha^4 - \alpha^3 - \alpha + (\alpha^2 + \alpha^6 + \alpha^3 + \alpha)\,\alpha^4$$
$$= -\alpha^4 - \alpha^3 - \alpha + \alpha^6 + \alpha^{10} + \alpha^7 + \alpha^5 = \alpha^6 + \alpha^5 - \alpha^4 - \alpha + 1,$$

$$b_0 = c_1 = \alpha^6 - \alpha^3 + 1,$$

$$c_0' = -\alpha^4 - \alpha^3 - \alpha + (\alpha^2 + \alpha^6 + \alpha^3 + \alpha)\,\alpha^6$$
$$= -\alpha^4 - \alpha^3 - \alpha + \alpha^8 + \alpha^{12} + \alpha^9 + \alpha^7 = \alpha^5 - \alpha^4 - \alpha^3 + \alpha^2 + 1.$$

From this it follows that $\Omega_0 = \Omega$ and that this point is the centre of the similarity transformation under which $\triangle\overrightarrow{C_2A_2B_2}$ goes into $\triangle\overrightarrow{B_1C_1A_1}$. That these triangles are similar and have opposite orientations follows from the fact that the construction of $\triangle\overrightarrow{C_2A_2B_2}$ on the basis of $\triangle\overrightarrow{B_1C_1A_1}$ is exactly the same as the construction of $\triangle\overrightarrow{B_1C_1A_1}$ on the basis of $\triangle\overrightarrow{ACB}$; the

figures under consideration with two circles circumscribed about $\triangle ABC$ and $\triangle A_1B_1C_1$ and all the corresponding triangles are samilar.

Furthermore, since $C_0A_2A_0B_0'$ is a rhombus, it follows that

$$a_2 - c_0 = a_0 - b_0',$$

whence

$$a_2 = a_0 + c_0 - b_0'$$
$$= \alpha^2 - \alpha^4 + \alpha^6 + \alpha^2 - \alpha + 1 - \alpha^6 - \alpha^5 + \alpha^4 + \alpha - 1 = 2\alpha^2 - \alpha^5.$$

Again, $B_2A_0C_0'B_0$ is a rhombus and so

$$a_0 - b_2 = c_0' - b_0,$$

whence

$$b_2 = a_0 + b_0 - c_0'$$
$$= \alpha^2 - \alpha^4 + \alpha^6 + \alpha^6 - \alpha^3 + 1 - \alpha^5 + \alpha^4 + \alpha^3 - \alpha^2 - 1 = 2\alpha^6 - \alpha^5.$$

Finally, since $C_2C_0A_0'B_0$ is a rhombus, it follows that

$$c_0 - c_2 = a_0' - b_0,$$

whence

$$c_2 = c_0 + b_0 - a_0'$$
$$= \alpha^2 - \alpha + 1 + \alpha^6 - \alpha^3 + 1 - \alpha^6 - \alpha^5 + \alpha^3 - \alpha^2 + \alpha = 2 - \alpha^5.$$

Thus,

$$a_2 = 2\alpha^2 - \alpha^5,$$
$$b_2 = 2\alpha^6 - \alpha^5,$$
$$c_2 = 2 - \alpha^5$$

and so $\triangle \overrightarrow{C_2B_2A_2}$ is obtained from the triangle $\overrightarrow{ABC}$ (the affixes of whose vertices are 1, α^6, α^2) via a homothetic transformation with centre Ω and ratio 2 since

$$\frac{c_2 - \omega}{a - \omega} = \frac{2 - \alpha^5 - \alpha^5}{1 - \alpha^5} = 2,$$

$$\frac{b_2 - \omega}{b - \omega} = \frac{2\alpha^6 - \alpha^5 - \alpha^5}{\alpha^6 - \alpha^5} = 2,$$

$$\frac{a_2 - \omega}{c - \omega} = \frac{2\alpha^2 - 2\alpha^5}{\alpha^2 - \alpha^5} = 2.$$

4°. The slope of the straight line $O\Omega$ is equal to

$$\frac{\alpha^5}{\bar{\alpha}^5} = \alpha^{10} = \alpha^3.$$

The straight line τ passes through the points A and B' and so its slope is

$$\frac{1-\alpha^3}{1-\bar{\alpha}^3}=\frac{1-\alpha^3}{1-\dfrac{1}{\alpha^3}}=-\alpha^3.$$

The sum of the slopes of the straight lines $O\Omega$ and τ is 0 and so $O\Omega \perp \tau$.

Problem 24. 1°. On the sides of the hexagon $A_1A_2A_3A_4A_5A_6$ are constructed equilateral triangles

$$\overrightarrow{A_1A_2A_1'},\ \overrightarrow{A_2A_3A_2'},\ \overrightarrow{A_3A_4A_3'},\ \overrightarrow{A_4A_5A_4'},\ \overrightarrow{A_5A_6A_5'},\ \overrightarrow{A_6A_1A_6'} \tag{66}$$

which have the same orientation. Prove that the ends P_1', P_3', P_5' of the directed line segments $\overrightarrow{OP_1'}$, $\overrightarrow{OP_3'}$, $\overrightarrow{OP_5'}$ (O is an arbitrary point), which are respectively equipollent to the directed line segments $\overrightarrow{A_1'A_4'}$, $\overrightarrow{A_3'A_6'}$, $\overrightarrow{A_5'A_2'}$, form an equilateral triangle $T = P_1'P_3'P_5'$, which has an orientation opposite that of any one of the triangles (66). In particular, $\triangle T$ can degenerate into a point.

2°. Prove the converse: namely, that if an arbitrary equilateral triangle $T = P_1'P_3'P_5'$ and a point O are chosen in a plane and directed line segments $\overrightarrow{A_1'A_4'}$, $\overrightarrow{A_3'A_6'}$, $\overrightarrow{A_5'A_2'}$ are constructed that are respectively equipollent to the directed line segments $\overrightarrow{OP_1'}$, $\overrightarrow{OP_3'}$, $\overrightarrow{OP_5'}$ (the position of the directed line segments $\overrightarrow{A_1'A_4'}$, $\overrightarrow{A_3'A_6'}$, $\overrightarrow{A_5'A_2'}$ are otherwise arbitrary), then, by choosing another arbitrary point A_1, it is possible to construct the points A_2, A_3, A_4, A_5, A_6 such that the hexagon $A_1'A_2'A_3'A_4'A_5'A_6'$ is obtained from the hexagon $A_1A_2A_3A_4A_5A_6$ by the construction indicated in item 1°. Prove that if the point A_1 is changed in the plane, then the principal diagonals A_1A_4, A_3A_6, A_5A_2 of the hexagon $A_1A_2A_3A_4A_5A_6$ (which will change together with any change in point A_1) will rotate about three points O_1, O_2, O_3. How are these points constructed?

3°. The equilateral triangles

$$\overrightarrow{A_1'A_2'A_1''},\ \overrightarrow{A_2'A_3'A_2''},\ \overrightarrow{A_3'A_4'A_3''},\ \overrightarrow{A_4'A_5'A_4''},\ \overrightarrow{A_5'A_6'A_5''},\ \overrightarrow{A_6'A_1'A_6''} \tag{67}$$

are constructed so that they all have the same orientation and the orientation of any one of them is opposite that of any one of the triangles (66). Prove that

$$\overrightarrow{A_1A_1''} \equiv \overrightarrow{A_2A_3},\ \overrightarrow{A_2A_2''} \equiv \overrightarrow{A_3A_4},\ \overrightarrow{A_3A_3''} \equiv \overrightarrow{A_4A_5},$$

$$\overrightarrow{A_4A_4''} \equiv \overrightarrow{A_5A_6},\ \overrightarrow{A_5A_5''} \equiv \overrightarrow{A_6A_1},\ \overrightarrow{A_6A_6''} \equiv \overrightarrow{A_1A_2}$$

where $\equiv$ is the sign of equipollency of (directed) line segments.

4°. Prove that

$$\overrightarrow{A_1''A_4''} \equiv \overrightarrow{A_3''A_6''} \equiv \overrightarrow{A_5''A_2''} \equiv 3\overrightarrow{OG},$$

where G is the centroid of the triangle $T^* = M_{14}M_{36}M_{52}$ and M_{14}, M_{36}, M_{52} are the ends of the directed line segments $\overrightarrow{OM_{14}}$, $\overrightarrow{OM_{36}}$, $\overrightarrow{OM_{52}}$ laid off from an arbitrary point O and respectively equipollent to the directed line segments $\overrightarrow{A_1A_4}$, $\overrightarrow{A_3A_6}$, $\overrightarrow{A_5A_2}$.

5°. Prove that the midpoint of the principal diagonal A_1A_4 of the hexagon $A_1A_2A_3A_4A_5A_6$ coincides with the point K_{14}, in which segments $A_1''A_2''$ and $A_4''A_5''$ intersect and are bisected; the midpoint of A_3A_6 coincides with the point K_{36}, in which segments $A_3''A_4''$ and $A_6''A_1''$ intersect and are bisected; finally, the midpoint of A_5A_2 coincides with the point K_{52}, in which segments $A_5''A_6''$ and $A_2''A_3''$ intersect and are bisected (Fig. 27).

Solution. We introduce in the plane a rectangular Cartesian system of coordinates Oxy. Denote by α and $\bar{\alpha}$ the imaginary roots of the equation $x^3 + 1 = 0$:

$$\alpha = \frac{1 + i\sqrt{3}}{2}, \quad \bar{\alpha} = \frac{1 - i\sqrt{3}}{2}.$$

Let a_1, a_2, a_3, a_4, a_5, a_6, p_1', p_3', p_5', a_1', a_2', a_3', a_4', a_5', a_6', be the respective affixes of the points A_1, A_2, A_3, A_4, A_5, A_6, P_1', P_3', P_5', A_1', A_2', A_3', A_4', A_5', A_6'. Then

$$\begin{aligned}
a_2 - a_1 &= \alpha(a_1' - a_1),\\
a_3 - a_2 &= \alpha(a_2' - a_2),\\
a_4 - a_3 &= \alpha(a_3' - a_3),\\
a_5 - a_4 &= \alpha(a_4' - a_4),\\
a_6 - a_5 &= \alpha(a_5' - a_5),\\
a_1 - a_6 &= \alpha(a_6' - a_6).
\end{aligned}$$

From this we find (note that since $\alpha^2 - \alpha + 1 = 0$ it follows that $\alpha - 1 + \bar{\alpha} = 0$, whence $1 - \alpha = \bar{\alpha}$)

$$\left.\begin{aligned}
a_2 &= \alpha a_1' + \bar{\alpha} a_1,\\
a_3 &= \alpha a_2' + \bar{\alpha} a_2,\\
a_4 &= \alpha a_3' + \bar{\alpha} a_3,\\
a_5 &= \alpha a_4' + \bar{\alpha} a_4,\\
a_6 &= \alpha a_5' + \bar{\alpha} a_5,\\
a_1 &= \alpha a_6' + \bar{\alpha} a_6.
\end{aligned}\right\} \tag{68}$$

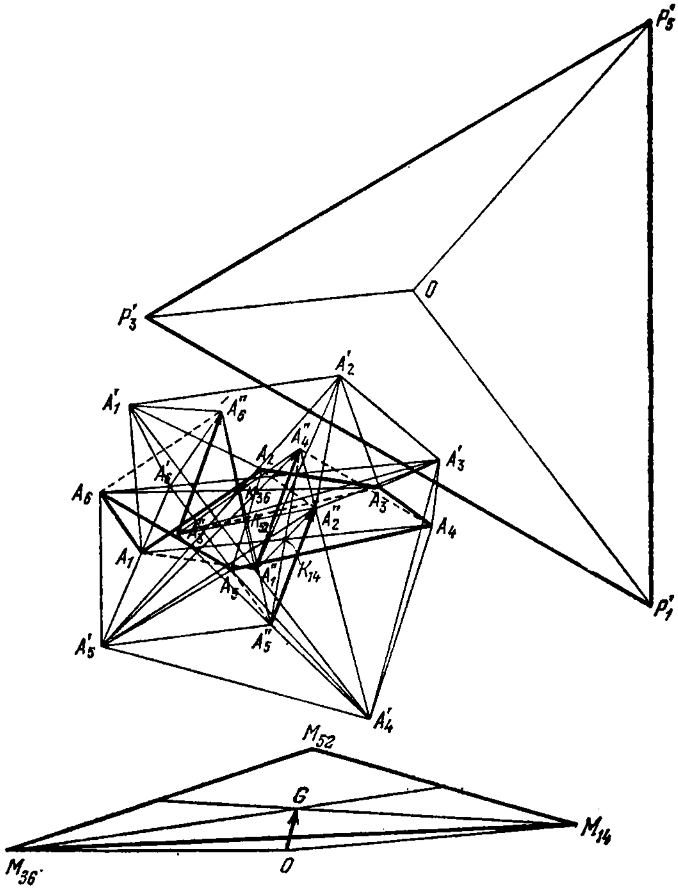

Fig. 27

Multiplying these relations by $\bar{\alpha}^5$, $\bar{\alpha}^4$, $\bar{\alpha}^3$, $\bar{\alpha}^2$, $\bar{\alpha}$, 1 respectively and adding, we obtain (note that $\alpha\bar{\alpha} = 1$, $\alpha^3 = \bar{\alpha}^3 = -1$)

$$a_2' - a_5' = \alpha(a_6' - a_3') + \bar{\alpha}(a_4' - a_1').$$

Since

$$a_4' - a_1' = p_1', \ a_6' - a_3' = p_3', \ a_2' - a_5' = p_5',$$

it follows that

$$p_5' = \alpha p_3' + \bar{\alpha} p_1'$$

or

$$p_5' = \alpha p_3' + (1 - \alpha) = p_1'$$

or

$$p_5' - p_1' = \alpha (p_3' - p_1'). \tag{69}$$

From this it follows that $\triangle P_1'P_3'P_5'$ (provided it does not degenerate into a point) is equilateral because from relation (69) it follows that the directed line segment $\overrightarrow{P_1'P_5'}$ is obtained by a rotation of the directed line segment $\overrightarrow{P_1'P_3'}$ through the angle $\pi/3$ ($\alpha = (1 + i\sqrt{3})/2 = \cos(\pi/3 + i\sin(\pi/3))$. The orientation of $\triangle \overrightarrow{P_1'P_3'P_5'}$ is opposite that of any one of the triangles of (66): this follows from relation (69) and, for example, from the relation

$$a_2 - a_1 = \alpha(a_1' - a_1);$$

since $\alpha\bar{\alpha} = 1$, it follows that

$$a_1' - a_1 = \bar{\alpha}(a_2 - a_1),$$

that is, the directed line segment $\overrightarrow{A_1A_1'}$ is obtained by a rotation of the directed line segment $\overrightarrow{A_1A_2}$ through the angle $-\pi/3$ ($\bar{\alpha} = \cos(-\pi/3) + i\sin(-\pi/3)$).

The case $p_1' = p_3'$ is also possible; then also $p_1' = p_5'$, that is, $\triangle P_1'P_3'P_5'$ contracts to a point. This occurs if and only if the directed principal diagonals $\overrightarrow{A_1'A_4'}$, $\overrightarrow{A_3'A_6'}$, $\overrightarrow{A_5'A_2'}$ of the hexagon $A_1'A_2'A_3'A_4'A_5'A_6'$ are equipollent:

$$\overrightarrow{A_1'A_4'} \equiv \overrightarrow{A_3'A_6'} \equiv \overrightarrow{A_5'A_2'}.$$

2°. Let $\overrightarrow{P_1'P_3'P_5'}$ be an arbitrary equilateral triangle and O an arbitrary point in the plane; let $\overrightarrow{A_1'A_4'}$, $\overrightarrow{A_3'A_6'}$, $\overrightarrow{A_5'A_2'}$ be arbitrary directed line segments respectively equipollent to the line segments $\overrightarrow{OP_1'}$, $\overrightarrow{OP_3'}$, $\overrightarrow{OP_5'}$:

$$\overrightarrow{OP_1'} \equiv \overrightarrow{A_1'A_4'},\ \overrightarrow{OP_3'} \equiv \overrightarrow{A_3'A_6'},\ \overrightarrow{OP_5'} \equiv \overrightarrow{A_5'A_2'} \tag{70}$$

Choose an arbitrary point A_1 in the plane. Construct the following equilateral triangles with the same orientation:

$$\overrightarrow{A_1A_2A_1'},\ \overrightarrow{A_2A_3A_2'},\ \overrightarrow{A_3A_4A_3'},\ \overrightarrow{A_4A_5A_4'},\ \overrightarrow{A_5A_6A_5'},\ \overrightarrow{A_6A_1A_6'}$$

but with orientation opposite that of $\triangle\overrightarrow{P_1'P_3'P_5'}$. On the basis of item 1°, with the aid of the construction given in item 1°, the hexagon $A_1A_2A_3A_4A_5A_6$ leads to the chosen points A_1', A_2', A_3', A_4', A_5', A_6' and then to the chosen triangle $P_1'P_3'P_5'$.

3°. The affixes $a_1', a_2', a_3', a_4', a_5', a_6'$ of the points A_1', A_2', A_3', A_4', A_5', A_6' are connected with the affixes $a_1'', a_2'', a_3'', a_4'', a_5'', a_6''$ of the points A_1'', A_2'',

A_3'', A_4'', A_5'', A_6'' by relations of the form (68) in which the positions of only α and $\bar{\alpha}$ need be interchanged:

$$a_2' = \bar{\alpha}a_1'' + \alpha a_1',$$
$$a_3' = \bar{\alpha}a_2'' + \alpha a_2',$$
$$a_4' = \bar{\alpha}a_3'' + \alpha a_3',$$
$$a_5' = \bar{\alpha}a_4'' + \alpha a_4',$$
$$a_6' = \bar{\alpha}a_5'' + \alpha a_5',$$
$$a_1' = \bar{\alpha}a_6'' + \alpha a_6'.$$

From these relations and from the relations (68) we find

$$\bar{\alpha}a_1'' = a_2' - \alpha a_1' = a_3\bar{\alpha} - a_2\bar{\alpha} + a_1\bar{\alpha} = \bar{\alpha}a_3 - \bar{\alpha}a_2 + \bar{\alpha}a_1$$

and so

Similarly

$$\left.\begin{aligned} a_1'' &= a_1 - a_2 + a_3. \\ a_2'' &= a_2 - a_3 + a_4, \\ a_3'' &= a_3 - a_4 + a_5, \\ a_4'' &= a_4 - a_5 + a_6, \\ a_5'' &= a_5 - a_6 + a_1, \\ a_6'' &= a_6 - a_1 + a_2. \end{aligned}\right\} \tag{71}$$

From these relations it follows that

$$a_1'' - a_1 = a_3 - a_2,$$
$$a_2'' - a_2 = a_4 - a_3,$$
$$a_3'' - a_3 = a_5 - a_4,$$
$$a_4'' - a_4 = a_6 - a_5,$$
$$a_5'' - a_5 = a_1 - a_6,$$
$$a_6'' - a_6 = a_2 - a_1,$$

and therefore

$$\overrightarrow{A_1A_1''} \equiv \overrightarrow{A_2A_3}, \quad \overrightarrow{A_2A_2''} \equiv \overrightarrow{A_3A_4}, \quad \overrightarrow{A_3A_3''} \equiv \overrightarrow{A_4A_5},$$
$$\overrightarrow{A_4A_4''} \equiv \overrightarrow{A_5A_6}, \quad \overrightarrow{A_5A_5''} \equiv \overrightarrow{A_6A_1}, \quad \overrightarrow{A_6A_6''} \equiv \overrightarrow{A_1A_2}.$$

4°. From the relations (71) it also follows that

$$a_4'' - a_1'' = a_6'' - a_3'' = a_2'' - a_5'' = a_4 - a_5 + a_6 - a_1 + a_2 - a_3. \tag{72}$$

Consequently,

$$\overrightarrow{A_1''A_4''} \equiv \overrightarrow{A_3''A_6''} \equiv \overrightarrow{A_5''A_2''}.$$

If we construct the directed line segments

$$\overrightarrow{OM_{14}} \equiv \overrightarrow{A_1A_4},\ \overrightarrow{OM_{36}} \equiv \overrightarrow{A_3A_6},\ \overrightarrow{OM_{52}} \equiv \overrightarrow{A_5A_2},$$

then the affixes m_{14}, m_{36}, m_{52} of the points M_{14}, M_{36}, M_{52} will be

$$m_{14} = a_4 - a_1,\ m_{36} = a_6 - a_3,\ m_{52} = a_2 - a_5$$

and so the affix g of the centroid of $\triangle M_{14}M_{36}M_{52}$ will be

$$g = (a_4 - a_1 + a_6 - a_3 + a_2 - a_5)/3.$$

From this and from the relations (72) it follows that

$$\overrightarrow{A_1''A_4''} \equiv \overrightarrow{A_3''A_6''} \equiv \overrightarrow{A_5''A_2''} \equiv \overrightarrow{3OG}.$$

5°. The midpoints of segments $A_1''A_2''$, $A_4''A_5''$, A_1A_4 have the same affix

$$\frac{a_1 + a_4}{2}$$

[see formulas (71)] and, hence, these midpoints coincide with the point K_{14}. The remaining propositions in item 5° can be proved in similar fashion.

Problem 25. Let A_1, B_1, C_1 be the orthogonal projections of the point P on the sides BC, CA, AB of triangle ABC. Let us construct $\triangle\ \overrightarrow{C_1B_1Q_1}$ similar to $\triangle\ \overrightarrow{BCP}$ but with opposite orientation. Let A', B', C' be the second points of intersection of the straight lines PA, PB, PC with the circle $(ABC) = (O)$.

1°. Prove that the triangles $\overrightarrow{A_1C_1Q_1}$ and $\overrightarrow{CAP}$ are similar and have opposite orientations.

2°. Prove that the triangles $\overrightarrow{B_1A_1Q_1}$ and $\overrightarrow{ABP}$ are similar and have opposite orientations.

3°. Prove that the triangles $\overrightarrow{A_1B_1C_1}$ and $\overrightarrow{A'B'C'}$ are similar and have the same orientation. Find the proportionality factor.

4°. Prove that the point Q_1 is a point obtained by inversion with the circle of inversion $(A_1B_1C_1)$ (or by symmetry with respect to the straight line $A_1B_1C_1$ for the case when the points A_1, B_1, C_1 lie on one straight line) of the midpoint T, of line segment PP^*, where P^* is the image of point P under inversion with the circle of inversion (ABC).

5°. Find the affix of the fixed point of the similarity transformation that carries one of the triangles $\overrightarrow{A_1B_1C_1}$ and $\overrightarrow{A'B'C'}$ into the other.

6°. For what position of point P does it coincide with point Q_1 (Fig. 28)?

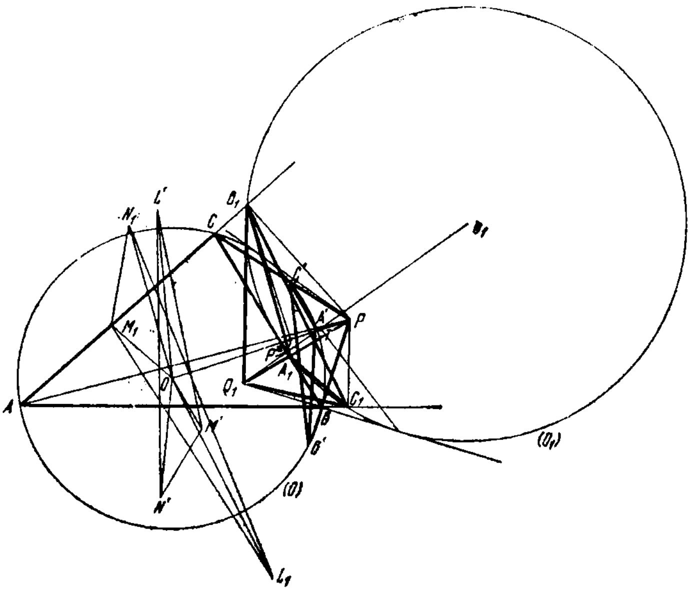

Fig. 28

Solution. 1°. Take $(O) = (ABC)$ for the unit circle. Let z_1, z_2, z_3 be the respective affixes of the points A, B, C. The equation of the straight line BC is of the form

$$z - z_2 = -z_2 z_3 (\bar{z} - \bar{z}_2)$$

or

$$z + z_2 z_3 \bar{z} = z_2 + z_3. \tag{73}$$

The equation of the perpendicular dropped from point P to line BC is

$$z - p = z_2 z_3 (\bar{z} - \bar{p})$$

or

$$z - z_2 z_3 \bar{z} = p - z_2 z_3 \bar{p}, \tag{74}$$

where p is the affix of point P.

Adding termwise the equations (73) and (74), we find the affix $z = a_1$ of point A_1:

$$a_1 = \frac{1}{2}(z_2 + z_3 + p - z_2 z_3 \bar{p}).$$

Similarly, we can find the affixes b_1 and c_1 of points B_1 and C_1:

$$b_1 = \frac{1}{2}(z_3 + z_1 + p - z_3 z_1 \bar{p}),$$

$$c_1 = \frac{1}{2}(z_1 + z_2 + p - z_1 z_2 \bar{p}).$$

Since it is given that $\triangle \overrightarrow{BCP}$ and $\triangle \overrightarrow{C_1 B_1 Q_1}$ are similar but have opposite orientations, it follows that the affix q_1 of point Q_1 can be found from the condition

$$\begin{vmatrix} \bar{b} & c_1 & 1 \\ \bar{c} & b_1 & 1 \\ \bar{p} & q_1 & 1 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} \dfrac{1}{z_2} & \dfrac{1}{2}(z_1 + z_2 + p - z_1 z_2 \bar{p}) & 1 \\ \dfrac{1}{z_3} & \dfrac{1}{2}(z_3 + z_1 + p - z_3 z_1 p) & 1 \\ \bar{p} & q_1 & 1 \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} \dfrac{1}{z_2} & z_1 + z_2 + p - z_1 z_2 \bar{p} & 1 \\ \dfrac{1}{z_3} & z_3 + z_1 + p - z_3 z_1 \bar{p} & 1 \\ \bar{p} & 2q_1 & 1 \end{vmatrix} = 0.$$

Subtract the second row from the first row of the determinant to get

$$\begin{vmatrix} \dfrac{z_3 - z_2}{z_2 z_3} & z_2 - z_3 + (z_3 - z_2)\,\bar{p} z_1 & 0 \\ \dfrac{1}{z_3} & z_3 + z_1 + p - z_3 z_1 \bar{p} & 1 \\ \bar{p} & 2q_1 & 1 \end{vmatrix} = 0,$$

Cancelling $z_3 - z_2 \neq 0$, we get

$$\frac{1}{z_2 z_3}(z_3 + z_1 + p - z_3 z_1 \bar{p} - 2q_1) - \left(\frac{1}{z_3} - \bar{p}\right)(z_1 \bar{p} - 1) = 0.$$

Multiplying the left-hand side by z_2z_3, we obtain

$$z_3 + z_1 + p - z_3z_1\bar{p} - 2q_1 - (z_2 - z_2z_3\bar{p})(z_1\bar{p} - 1) = 0$$

or

$$z_3 + z_1 + p - z_3z_1\bar{p} - 2q_1 - z_2z_1\bar{p} + z_2 - \sigma_3\bar{p}^2 - z_2z_3p = 0$$

whence

$$q_1 = \frac{1}{2}(\sigma_3\bar{p}^2 - \sigma_2p + p + \sigma_1).$$

2°. The symmetry of the right-hand side with respect to z_1, z_3, z_3 permits asserting that (item 1°) $\triangle\ \overrightarrow{A_1C_1Q_1}$ and $\triangle\ \overrightarrow{CAP}$ are similar and have opposite orientations and that (item 2°) $\triangle\ \overrightarrow{B_1A_1Q_1}$ and $\triangle\ \overrightarrow{ABP}$ are similar and have opposite orientations. At the same time we found the affix q_1 of point Q_1.

3°. Associated with the directed line segment $\overrightarrow{A_1B_1}$ is the complex number

$$b_1 - a_1 = \frac{1}{2}[z_1 - z_2 + z_3(z_2 - z_1)\bar{p}] = \frac{1}{2}(z_2 - z_1)(z_3\bar{p} - 1).$$

Let us find the affixes a', b', c' of the points A', B', C'. The equation of the straight line PA is of the form

$$z - z_1 = \frac{p - z_1}{\bar{p} - \bar{z}_1}(\bar{z} - \bar{z}_1).$$

Solving this equation together with the equation $z\bar{z} = 1$ of the unit circle (ABC), we obtain

$$z - z_1 = \frac{p - z_1}{p - p_1}\left(\frac{1}{z} - \frac{1}{z_1}\right),$$

$$z - z_1 = -\frac{p - z_1}{p - \bar{z}_1}\,\frac{z - z_1}{zz_1}.$$

One of the roots of this equation is naturally $z = z_1$ (the affix of point A); the other root (the affix a' of point A') and be found from the equation

$$1 = -\frac{p - z_1}{\bar{p} - \bar{z}_1}\,\frac{1}{zz_1},$$

whence

$$z = a' = -\frac{1}{z_1}\,\frac{p - z_1}{\bar{p} - \bar{z}_1}$$

and b' and c' have similar expressions. Thus,

$$a' = -\frac{1}{z_1}\frac{p - z_1}{\bar{p} - \bar{z}_1},$$

$$b' = -\frac{1}{z_2}\frac{p - z_2}{\bar{p} - \bar{z}_2},$$

$$c' = -\frac{1}{z_3}\frac{p - z_3}{\bar{p} - \bar{z}_3}.$$

To the directed line segment $\overrightarrow{A'B'}$ there corresponds the complex number

$$b' - a' = \frac{1}{z_1}\frac{p - z_1}{\bar{p} - \bar{z}_1} - \frac{1}{z_2}\frac{p - z_2}{\bar{p} - \bar{z}_2}$$

$$= \frac{z_2(p\bar{p} - z_1\bar{p} - \bar{z}_2 p + z_1\bar{z}_2) - z_1(p\bar{p} - \bar{z}_1 p - z_2\bar{p} + \bar{z}_1 z_2)}{z_1 z_2(\bar{p} - \bar{z}_1)(\bar{p} - \bar{z}_2)}$$

$$= \frac{z_2\bar{p}p - z_1 z_2\bar{p} - p + z_1 - z_1 p\bar{p} + p + z_1 z_2\bar{p} - z_2}{z_1 z_2(\bar{p} - \bar{z}_1)(\bar{p} - \bar{z}_2)}$$

$$= \frac{p\bar{p}(z_2 - z_1) - (z_2 - z_1)}{z_1 z_2(\bar{p} - \bar{z}_1)(\bar{p} - \bar{z}_2)} = \frac{(z_2 - z_1)(p\bar{p} - 1)}{z_1 z_2(\bar{p} - \bar{z}_1)(\bar{p} - \bar{z}_2)}.$$

From this it follows that

$$\frac{b' - a'}{b_1 - a_1} = \frac{2(p\bar{p} - 1)}{z_1 z_2(\bar{p} - \bar{z}_1)(\bar{p} - \bar{z}_2)(z_3\bar{p} - 1)}$$

$$= \frac{2(p\bar{p} - 1)}{\sigma_3(\bar{p} - \bar{z}_1)(\bar{p} - \bar{z}_2)(\bar{p} - \bar{z}_3)} = \frac{2(p\bar{p} - 1)}{\sigma_3(\bar{p}^3 - \bar{\sigma}_1\bar{p}^2 + \bar{\sigma}_2\bar{p} - \bar{\sigma}_3)}; \quad (75)$$

and since $\bar{\sigma}_1 = \frac{\sigma_2}{\sigma_3}$, $\bar{\sigma}_2 = \frac{\sigma_1}{\sigma_3}$, it follows that

$$\frac{b' - a'}{b_1 - a_1} = \frac{2(p\bar{p} - 1)}{\sigma_3\bar{p}^3 - \sigma_2\bar{p}^2 + \sigma_1\bar{p} - 1} \quad (76)$$

Denoting the right-hand side by λ, we have

$$\lambda = \frac{2(p\bar{p} - 1)}{\sigma_3\bar{p}^3 - \sigma_2\bar{p}^2 + \sigma_1\bar{p} - 1}$$

and then obtain

$$b' - a' = \lambda(b_1 - a_1)$$

and, similarly (since λ is a symmetric function of z_1, z_2, z_3),

$$c' - b' = \lambda(c_1 - b_1),$$
$$a' - c' = \lambda(a_1 - c_1).$$

Consequently, $\triangle \overrightarrow{A'B'C'}$ and $\triangle \overrightarrow{A_1B_1C_1}$ are similar and have the same orientation.

On the basis of (75), the proportionality factor may be written as

$$|\lambda| = \frac{2|OP^2 - R^2|}{|\bar{p} - \bar{z}_1|\,|\bar{p} - \bar{z}_2|\,|\bar{p} - \bar{z}_3|} = \frac{2|OP^2 - R^2|}{PA \cdot PB \cdot PC}.$$

4°. Let us consider the similarity transformation that carries $\triangle \overrightarrow{A_1B_1C_1}$ into $\triangle \overrightarrow{A'B'C'}$. Under this transformation, the centre O_1 of the circle $(A_1B_1C_1)$ goes into the centre $O' \equiv O$ of the circle $(A'B'C')$, and the point H_1, the point of intersection of the altitudes of $\triangle A_1B_1C_1$, goes into H', the point of intersection of the altitudes of $\triangle A'B'C'$. Since the indicated similarity transformation is of the form

$$z' = \lambda z_1 + \mu$$

and since under this transformation the directed line segment $\overrightarrow{O_1A_1}$ goes into the directed line segment $\overrightarrow{O'A'} \equiv \overrightarrow{OA'}$, it follows that

$$a' = \lambda(a_1 - o_1), \tag{77}$$

where a', a_1, o_1 are the respective affixes of the points A', A_1, O_1. Similarly,

$$\left.\begin{aligned} b' &= \lambda(b_1 - o_1),\\ c' &= \lambda(c_1 - o_1),\end{aligned}\right\} \tag{78}$$

where b', c', b_1, c_1 are the respective affixes of the points B', C', B_1, C_1 From (77) and (78) we find

$$a' + b' + c' = \lambda(a_1 + b_1 + c_1 - 3o_1).$$

Now, using the earlier obtained expressions for a', b', c', we have

$$a' + b' + c' = -\frac{1}{z_1}\frac{p - z_1}{\bar{p} - \bar{z}_1} - \frac{1}{z_2}\frac{p - z_2}{\bar{p} - \bar{z}_2} - \frac{1}{z_3}\frac{p - z_3}{\bar{p} - \bar{z}_3} = -\frac{1}{\sigma^3}\frac{X}{Y}, \tag{79}$$

where

$$X = z_2z_3(p - z_1)(\bar{p} - \bar{z}_2)(\bar{p} - \bar{z}_3) + z_3z_1(p - z_2)(\bar{p} - \bar{z}_3)(\bar{p} + \bar{z}_1) + z_1z_2(p - z_3)(\bar{p} - \bar{z}_1)(\bar{p} - \bar{z}_2),$$
$$Y = (\bar{p} - \bar{z}_1)(\bar{p} - \bar{z}_2)(\bar{p} - \bar{z}_3).$$

Furthermore,

$$z_2z_3(\bar{p} - \bar{z}_2)(\bar{p} - \bar{z}_3)(p - z_1) = z_2z_3[\bar{p}^2 - (\bar{z}_2 + \bar{z}_3)\bar{p} + \bar{z}_2\bar{z}_3](p - z_1)$$
$$= [z_2z_3\,\bar{p}^2 - (z_2 + z_3)\,\bar{p} + 1](p - z_1)$$
$$= z_2z_3\bar{p}^2p - (z_2 + z_3)\,p\bar{p} + p - \sigma_3\bar{p}^2 + (z_1z_2 + z_1z_3)\,\bar{p} - z_1.$$

The other two terms in the numerator of (79) are of the form

$$z_3z_1\,\bar{p}^2p - (z_3 + z_1)\,p\bar{p} + p - \sigma_3\bar{p}^2 + (z_2z_3 + z_2z_1)\,\bar{p} - z_2,$$
$$z_1z_2\bar{p}^2p - (z_1 + z_2)\,p\bar{p} + p - \sigma_3\bar{p}^2 + (z_3z_1 + z_3z_2)\,\bar{p} - z_3.$$

Adding the last three expressions, we obtain

$$X = p\bar{p}^2\sigma_3 - 2\sigma_1p\bar{p} + 3p - 3\sigma_3\bar{p}^2 - 2\sigma_2\bar{p} - \sigma_1.$$

To summarize:

$$a' + b' + c' = \frac{-p\bar{p}^2\sigma_2 + 2\sigma_1p\bar{p} - 3p + 3\sigma_3\bar{p}^2 - 2\sigma_2\bar{p} + \sigma_1}{\sigma_3(\bar{p} - \bar{z}_1)(\bar{p} - \bar{z}_2)(\bar{p} - \bar{z}_3)}.$$

Using the expressions for a_1, b_1, c_1 obtained above, we get

$$a_1 + b_1 + c_1 = \frac{1}{2}\,(2\sigma_1 + 3p - \sigma_2\bar{p});$$

and since $\lambda = \dfrac{2(p\bar{p} - 1)}{\sigma_3(\bar{p} - \bar{z}_1)(\bar{p} - \bar{z}_2)(\bar{p} - \bar{z}_3)}$, it follows that the relation

$$a' + b' + c' = \lambda\,(a_1 + b_1 + c_1 - 3o_1)$$

or

$$-\,3o_1 = \frac{1}{\lambda}\,(a' + b' + c') - (a_1 + b_1 + c_1)$$

may be rewritten thus:

$$-\,6o_1 = \frac{-p\bar{p}^2\sigma_2 + 2\sigma_1p\bar{p} - 3p + 3\sigma_3\bar{p}^2 - 2\sigma_2\bar{p} + \sigma_1}{p\bar{p} - 1} - (2\sigma_1 + 3p - \sigma_2\bar{p})$$
$$= \frac{-\,3p^2\bar{p} + 3\sigma_3\bar{p}^2 - 3\sigma_2\bar{p} + 3\sigma_1}{p\bar{p} - 1};$$

hence,

$$o_1 = \frac{p^2\bar{p} - \sigma_3\bar{p}^2 + \sigma_2\bar{p} - \sigma_1}{2(p\bar{p} - 1)}.$$

From this we obtain

$$q_1 - o_1 = \frac{\sigma_3\bar{p}^2 - \sigma_2\bar{p} + p + \sigma_1}{2} - \frac{p^2\bar{p} - \sigma_3\bar{p}^2 + \sigma_2\bar{p} - \sigma_1}{2(p\bar{p} - 1)}$$
$$= \frac{\sigma_3p\bar{p}^3 - \sigma_2p\bar{p}^2 + p^2\bar{p} + \sigma_1p\bar{p} - \sigma_3\bar{p}^2 + \sigma_2\bar{p} - p - \sigma_1 - p^2\bar{p} + \sigma_3\bar{p}^2 - \sigma_2\bar{p} + \sigma_1}{2(p\bar{p} - 1)} =$$

$$= \frac{\sigma_3 p\bar{p}^3 - \sigma_2 p\bar{p}^2 + \sigma_1 p\bar{p} - p}{2(p\bar{p} - 1)} = \frac{\sigma_3 p(\bar{p}^3 - \bar{\sigma}_1\bar{p}^2 + \bar{\sigma}_2\bar{p} - \bar{\sigma}_3)}{2(p\bar{p} - 1)}$$

$$= \frac{\sigma_3 p(\bar{p} - \bar{z}_1)(\bar{p} - \bar{z}_2)(\bar{p} - \bar{z}_3)}{2(p\bar{p} - 1)}.$$

The affix t of the midpoint T of segment PP^* is

$$t = \frac{1}{2}\left(p + \frac{1}{\bar{p}}\right) = \frac{p\bar{p} + 1}{2\bar{p}}.$$

We have

$$t - o_1 = \frac{p\bar{p} + 1}{2\bar{p}} - \frac{p^2\bar{p} - \sigma_3\bar{p}^2 + \sigma_2\bar{p} - \sigma_1}{2(p\bar{p} - 1)}$$

$$= \frac{p^2\bar{p}^2 - 1 - p^2\bar{p}^2 + \sigma_3\bar{p}^3 - \sigma_2\bar{p}^2 + \sigma_1\bar{p}}{2\bar{p}(p\bar{p} - 1)} = \frac{\sigma_3\bar{p}^3 - \sigma_2\bar{p}^2 + \sigma_1\bar{p} - 1}{2\bar{p}(p\bar{p} - 1)}$$

$$= \frac{\sigma_3(\bar{p} - \bar{z}_1)(\bar{p} - \bar{z}_2)(\bar{p} - \bar{z}_3)}{2\bar{p}(p\bar{p} - 1)},$$

whence

$$\overline{t - o_1} = \frac{\bar{\sigma}_3(p - z_1)(p - z_2)(p - z_3)}{2p(p\bar{p} - 1)}$$

and therefore

$$(q_1 - o_1)\overline{(t_1 - o_1)} = \frac{|p - z_1|^2\,|p - z_2|^2\,|p - z_3|^2}{4(p\bar{p} - 1)^2}.$$

But this means that the points T and Q_1 are obtained one from the other by inversion with the circle of inversion with centre O_1 and radius

$$\rho = \frac{|p - z_1|\,|p - z_2|\,|p - z_3|}{2\,|p\bar{p} - 1|} = \frac{PA\cdot PB\cdot PC}{2\,|OP^2 - R^2|} \quad (R = 1).$$

The radius of the circle $(A_1B_1C_1) = (O_1)$ can be found from the similarity relation $z' = \lambda z_1 = \mu$ that carries the circle $(A_1B_1C_1)$ into the circle $(A'\,B'\,C')$. Since the radius of $(A'B'C')$ is $R = 1$, it follows that the radius R_1 of $(A_1B_1C_1)$ is

$$R_1 = \frac{R}{|\lambda|} = \left|\frac{\sigma_3(\bar{p} - \bar{z}_1)(\bar{p} - \bar{z}_2)(\bar{p} - \bar{z}_3)}{2(p\bar{p} - 1)}\right| = \frac{PA\cdot PB\cdot PC}{2\,|OP^2 - R^2|} = \rho.$$

5°. The affix ω of the fixed point Ω of the similarity transformation that carries one of the triangles, $A_1B_1C_1$, into the other, $A'\,B'\,C'$, is found from

the reasoning that under this transformation $\triangle O_1A_1\Omega$ goes into $\triangle OA'\Omega$ so that

$$\begin{vmatrix} 0 & o_1 & 1 \\ a' & a_1 & 1 \\ \omega & \omega & 1 \end{vmatrix} = 0,$$

whence

$$\omega = \frac{o_1 a'}{o_i + a' - a_1}.$$

Furthermore,

$$a' - a_1 = \frac{z_1 - p}{z_1 p - 1} - \frac{z_2 - z_3 + p - z_2 z_3 \bar{p}}{2}$$

$$= \frac{2z_1 - 2p - z_1 z_2 \bar{p} - z_1 z_3 \bar{p} - z_1 p\bar{p} + \sigma_3 \bar{p}^2 + z_2 + z_3 + p - z_2 z_3 \bar{p}}{2(z_1 \bar{p} - 1)}$$

$$= \frac{z_1 - p + \sigma_1 - \sigma_3 \bar{p} - z_1 p \bar{p} + \sigma_3 p^2}{2(z_1 p - 1)},$$

$$o_1 + a' - a_1 = \frac{p^2 \bar{p} - \sigma_3 \bar{p}^2 + \sigma_2 \bar{p} - \sigma_1}{2(p\bar{p} - 1)} + \frac{z_1 - p + \sigma_1 - \sigma_2 \bar{p} - z_1 p \bar{p} + \sigma_3 \bar{p}^2}{2(z_1 \bar{p} - 1)}$$

$$= \frac{1}{2(p\bar{p} - 1)(z_1 \bar{p} - 1)} (p^2 \bar{p}^2 z_1 - \sigma_3 \bar{p}^3 z_1 + \sigma_2 z_1 \bar{p}^2 - \sigma_1 z_1 \bar{p} - p^2 \bar{p}$$

$$+ \sigma_3 \bar{p}^2 - \sigma_2 \bar{p} + \sigma_1 + p\bar{p} z_1 - p^2 \bar{p} + p\bar{p}\sigma_1 - \bar{p}^2 p \sigma_2 - z_1 p^2 \bar{p}^2$$

$$+ \sigma_3 p \bar{p}^3 - z_1 + p - \sigma_1 + \bar{p}\sigma_2 + z_1 p\bar{p} - \sigma_3 \bar{p}^2)$$

$$= \frac{2p\bar{p}(z_1 - p) - (z_1 - p) + \sigma_2 \bar{p}^2 (z_1 - p) - \sigma_1 \bar{p}(z_1 - p) - \sigma_3 p^3 (z_1 - p)}{2(p\bar{p} - 1)(z_1 \bar{p} - 1)}$$

$$= \frac{(z_1 - p)(2p\bar{p} - 1 + \sigma_2 \bar{p}^2 - \sigma_1 \bar{p} - \sigma_3 \bar{p}^3)}{2(p\bar{p} - 1)(z_1 \bar{p} - 1)}$$

and, consequently,

$$\omega = \frac{\dfrac{p^2 \bar{p} - \sigma_3 \bar{p}^2 + \sigma_2 \bar{p} - \sigma_1}{2(p\bar{p} - 1)} \cdot \dfrac{z_1 - p}{z_1 \bar{p} - 1}}{\dfrac{(z_1 - p)(2p\bar{p} - 1 + \sigma_2 \bar{p}^2 - \sigma_1 \bar{p} - \sigma_3 \bar{p}^3)}{2(p\bar{p} - 1)(z_1 \bar{p} - 1)}}$$

$$= \frac{p^2 \bar{p} - \sigma_3 \bar{p}^2 + \sigma_2 \bar{p} - \sigma_1}{2(p\bar{p} - 1) - (\sigma_3 \bar{p}^3 - \sigma_2 \bar{p}^2 + \sigma_1 \bar{p} - 1)}.$$

But

$$p^2\bar{p} - \sigma_3\bar{p}^2 + \sigma_2\bar{p} - \sigma_1 = 2o_1(p\bar{p} - 1),$$

$$\sigma_3\bar{p}^3 - \sigma_2\bar{p}^2 + \sigma_1\bar{p} - 1 = 2\bar{p}(p\bar{p} - 1)(t - o_1)$$

and so

$$\omega = \frac{2o_1(p\bar{p} - 1)}{2(p\bar{p} - 1) - 2\bar{p}(p\bar{p} - 1)(t - o_1)} = \frac{o_1}{1 - \bar{p}(t - o_1)}.$$

6°. The points P and Q_1 coincide if and only if $p = q_1$, that is,

$$p = \frac{1}{2}(\sigma_3\bar{p}^2 - \sigma_2\bar{p} + p + \sigma_1)$$

or

$$\sigma_3\bar{p}^2 - \sigma_2\bar{p} - p + \sigma_1 = 0. \tag{80}$$

From this,

$$\bar{\sigma}_3 p^2 - \bar{\sigma}_2 p - \bar{p} - \bar{\sigma}_1 = 0$$

or

$$\frac{1}{\sigma_3}p^2 - \frac{\sigma_1}{\sigma_3}p - \bar{p} + \frac{\sigma_2}{\sigma_3} = 0,$$

that is,

$$\bar{p} = \frac{p^2 - \sigma_1 p + \sigma_2}{\sigma_3}$$

and (80) takes the form

$$\frac{1}{\sigma_3}(p^4 + \sigma_1^2 p^2 + \sigma_2^2 - 2\sigma_1 p^3 + 2p^2\sigma_2 - 2\sigma_1\sigma_2 p)$$

$$- \frac{1}{\sigma_3}(\sigma_2 p^2 - \sigma_1\sigma_2 p + \sigma_2^2) - p + \sigma_1 = 0$$

or

$$p^4 - 2\sigma_1 p^3 + (\sigma_1^2 + \sigma_2)p^2 - (\sigma_1\sigma_2 + \sigma_3)p + \sigma_1\sigma_3 = 0. \tag{81}$$

It is clear that $p = \sigma_1$ is a root of this equation.

Dividing the polynomial in the left-hand side of (81) by $p - \sigma_1$, we obtain $p^2 - \sigma_1 p^2 + \sigma_2 p - \sigma_3 = 0$ or $(p - z_1)(p - z_2)(p - z_3) = 0$. Thus, the remaining roots of equaton (81) are

$$p = z_1, \ p = z_2, \ p = z_3.$$

Assuming that P does not coincide with any of the vertices of the given triangle, we conclude that the only possibility is $p = \sigma_1$. Thus, the points P

and Q_1 coincide if and only if P is the orthocentre of the given triangle ABC.

Problem 26. Take an arbitrary point P on a circle $(ABC) = (O)$. Let A^* be the point symmetric to point A with respect to the straight line OP, and let A' be the point symmetric to P with respect to the straight line OA^*. Construct points B' and C' in similar fashion. Denote by A'', B'', C'' points respectively symmetric to the points A', B', C' with respect to lines BC, CA, AB.

1°. Prove that the triangles $\overrightarrow{ABC}$ and $\overrightarrow{A''B''C''}$ are similar and have the same orientation.

2°. Prove that $\triangle ABC$ is inscribed in $\triangle A''B''C''$; namely, that the triplets of points A, B'', C''; B, C'', A''; C, A'', B'' lie on one straight line.

3°. Prove that the orthocentre H of $\triangle ABC$ is the centre of the circle $(A''B''C'')$.

4°. Find the radius R'' of $(A''B''C'')$ and indicate a method for constructing it if we know the position of the unit point on the unit circle (ABC) and the position of point P on that circle.

5°. What is the maximum value of R''? What position of point P is associated with this maximum value of R''?

6°. For what positions of the point P do all the points A'', B'', C'' coincide with the point H?

7°. For how many and what positions of point P are the triangles $A''B''C''$ and ABC congruent?

8°. Find the affix of the centre of the similarity transformation of $\triangle ABC$ and $\triangle A''B''C''$, assuming (ABC) to be the unit circle and knowing the affixes a, b, c, p of the points A, B, C, P.

9°. Prove that if the point P describes the unit circle (ABC), then the centre Ω' of the similarity transformation of $\triangle ABC$ and $\triangle A''B''C''$ describes the orthocentroidal circle of $\triangle ABC$, that is, the circle (GH) constructed on the line segment GH as a diameter (H is the orthocentre of $\triangle ABC$, G is its centroid, hence the term *orthocentroidal*).

10°. For how many and what positions of the point P do the points G and Ω' coincide?

11°. For how many and what positions of the point P do the points H and Ω' coincide?

12°. What line does the centroid G'' of $\triangle A''B''C''$ describe if point P describes the circle (ABC)?

13°. Prove that the affix of the centre Ω' of the similarity transformation of $\triangle ABC$ and $\triangle A''B''C''$ is connected with the affix of the centroid G'' of $\triangle A''B''C''$ by a linear fractional relation, which, consequently, determines a certain circular transformation of the plane. Prove that this transformation leaves in place the orthocentroidal circle (GH) of the triangle ABC. Find the fixed points of this transformation.

14°. Prove that the centroid G'' of $\triangle A''B''C''$, which centroid corresponds to the centre Ω_1' of similarity of $\triangle ABC$ and $\triangle A''B''C''$, is the

second point of intersection with the circle (GH) of the straight line $O_1\Omega^*$, where O_1 is a point symmetric to the point O about the midpoint K of segment GH, and Ω^* is a point symmetric to the point Ω' about the mid-perpendicular of segment GH.

Solution. 1°. We take (ABC) for the unit circle, and Boutain point of $\triangle ABC$ for the unit point. Let $a, b, c, p, a', b', c', a'', b'', c''$ be the respective affixes of the points $A, B, C, P, A', B', C', A'', B'', C''$. Then $\sigma_3 = abc = 1$. The equation of the straight line passing through point A and perpendicular to the straight line OP is of the form

$$z - a = -p^2(\bar{z} - \bar{a}).$$

Solving this equation together with the equation of the unit circle $z\bar{z} = 1$, we get

$$z - a = p^2\,\frac{z - a}{az}.$$

One of the roots of this equation, $z = a$, is the affix of point A, the other root is

$$z = a^* = p^2/a,$$

which is the affix of point A^*.

The slope of the straight line perpendicular to the straight line OA^* is

$$z - p = \varkappa(\bar{z} - \bar{p}),\quad -\frac{p^2}{a}\Big/\frac{\bar{p}^2}{\bar{a}} = -\frac{p^4}{a^2}.$$

The equation of the straight line passing through point P perpendicularly to OA^* is of the form

$$z - p = -\frac{p^4}{a^2}(\bar{z} - \bar{p}).$$

Solving this equation together with the equation of the unit circle $z\bar{z} = 1$, we find the affix a' of point A':

$$z - p = \frac{p^3}{a^2}\,\frac{z - p}{z}.$$

One of the roots of this equation is naturally $z = p$ (the affix of point P), the other is the affix of the point A', that is, $a' = p^3/a^2$. In similar fashion we find the affixes of the points B' and C'. We thus have

$$a' = \frac{p^3}{a^2},\quad b' = \frac{p^3}{b^2},\quad c' = \frac{p^3}{c^2}.$$

The equation of BC is

$$z + bc\bar{z} = b + c.$$

The equation of the perpendicular dropped from point A' to the straight line BC is

$$z - \frac{p^3}{a^2} = bc\left(\bar{z} - \frac{\bar{p}^3}{\bar{a}^2}\right)$$

or

$$z - bc\bar{z} = \frac{p^3}{a^2} - bc\frac{a^2}{p^3}.$$

Adding these last equations and the equation of BC termwise, we find the affix $z = a_*''$ of the projection of point A' on line BC:

$$a_*'' = \frac{1}{2}\left(b + c + \frac{p^3}{a^2} - a\frac{\sigma_3}{p_3}\right).$$

We find the affix a'' of point A'' from the relation

$$\frac{a' + a''}{2} = a_*''.$$

That is,

$$\frac{1}{2}\left(\frac{p^3}{a^2} + a''\right) = \frac{1}{2}\left(b + c + \frac{p^3}{a^2} - a\frac{\sigma_3}{p^3}\right),$$

whence (since $\sigma_3 = 1$)

$$a'' = b + c - a\frac{\sigma_3}{p^3} = \sigma_1 - a(1 + p^{-3}).$$

In similar fashion we find b'' and c''. Thus,

$$\left.\begin{aligned} a'' &= \sigma_1 - (1 + p^{-3})a, \\ b'' &= \sigma_1 - (1 + p^{-3})b, \\ c'' &= \sigma_1 - (1 + p^{-3})c. \end{aligned}\right\} \tag{82}$$

And so $\triangle\, A''B''C''$ is obtained from $\triangle\, ABC$ by a similarity transformation of the first kind:

$$z'' = \sigma_1 - (1 + p^{-3})z. \tag{83}$$

Hence the oriented triangles $\overrightarrow{ABC}$ and $\overrightarrow{A''B''C''}$ are similar and have the same orientation.

2°. We now prove that $\triangle ABC$ is inscribed in $\triangle\, A''B''C''$. We will prove that the straight line $B''C''$ passes through point A. To do this it suffices, for example, to prove that

$$\frac{a - b''}{a - c''}$$

is a real number. Taking advantage of formulas (82), we find

$$\frac{a-b''}{a-c''}=\frac{a-\sigma_1+b(1+p^{-3})}{a-\sigma_1+c(1+p^{-3})}=\frac{bp^{-3}-c}{cp^{-3}-b}=u.$$

We have

$$\bar{u}=\frac{\frac{1}{b}p^3-\frac{1}{c}}{\frac{1}{c}p^3-\frac{1}{b}}=\frac{cp^3-b}{bp^3-c}=\frac{bp^{-3}-c}{cp^{-3}-b}=u.$$

In a similar manner, proof is given that line $C''A''$ passes through point B and line $A''B''$ passes through point C.

3°. The image of point O under the similarity transformation (83) is a point with affix σ_1, that is, the orthocentre H of $\triangle\ ABC$. Under the similarity transformation (83), $\triangle\ ABC$ goes into $\triangle A''B''C''$ and so the centre O of (ABC) goes into the centre of $(A''B''C'')$, that is, the point H.

4°. The radius R'' of $(A''B''C'')$ is equal to $|1+p^{-3}|=|1+p^3|$ and, hence, may be constructed as follows (Fig.29): draw a tangent line to the circle (O) at the point P and through the unit point Ω (a Boutain point); draw the straight line l parallel to that tangent line; the second point of intersection, point Q, of the line l with the unit circle will have an affix p^2. Through point Ω draw a straight line m parallel to line PQ, the second point of intersection, point T, of line m and the unit circle will have the affix p^3. Construct a rhombus with sides $O\Omega$ and OT. Its diagonal $OS=|1+p^3|=R''$, where R'' is the radius of $(A''B''C'')$.

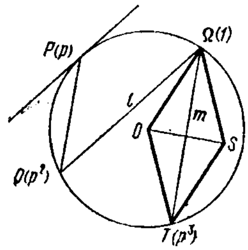

Fig. 29

5°. Let $p=\cos\varphi+i\sin\varphi$, then $1+p^3=1+\cos\ 3\varphi+i\sin\ 3\varphi$ and, hence,

$$R''=|1+p^3|=\sqrt{(1+\cos 3\varphi)^2+\sin^2\varphi}=2\,|\cos\ (3\varphi/2)|.$$

From this it follows that R'' is a maximum [twice the radius R of (ABC)] if and only if $|\cos\ (3\varphi/2)|=1$, whence $3\varphi/2=\pm k\pi$ and, consequently, $\varphi=\pm\ 2k\ \pi/3$. On the unit circle there are only three points T_1, T_3, T_5, which correspond, for example, to the following values of φ:

$$\varphi_1=0,\quad \varphi_3=2\pi/3,\quad \varphi_5=4\,\pi/3.$$

For $\varphi_1=0$ we obtain the unit point, that is, the Boutain point T_1 of $\triangle\ ABC$; T_3 and T_5 are the other two Boutain points [$\triangle\ T_1T_3T_5$ is an equilateral triangle inscribed in the circle (ABC)]. Thus, R'' assumes the maximum value, equal to $R''=2=2R$ if and only if the point P coincides with one of the three Boutain points of $\triangle\ ABC$.

6°. The points A'', B'', C'' coincide with the point H if and only if $p^3 + 1 = 0$, that is, $p = \sqrt[3]{-1}$. The affixes of these points T_2, T_4, T_6 have the arguments

$$\varphi_2 = \pi/3, \quad \varphi_4 = \pi, \quad \varphi_6 = 5\pi/3.$$

These points T_2, T_4, T_6 together with the Boutain points of $\triangle ABC$ form a regular hexagon $T_1T_2T_3T_4T_5T_6$ inscribed in the circle (ABC). Thus, all the points A'', B'', C'' coincide if and only if the point P coincides with one of the three vertices T_2, T_4, T_6 of a regular hexagon inscribed in the circle (ABC), whose three other vertices are the Boutain points T_1, T_3, T_5 of $\triangle ABC$.

7°. The triangles ABC and $A''B''C''$ are congruent under six positions of the point P. Indeed, $\triangle ABC$ and $\triangle A''B''C''$ are congruent if and only if $|p^3 + 1| = 1$, that is, $2 |\cos (3\varphi/2)| = 1$, whence

$$\cos (3\varphi/2) = \pm 1/2, \quad 3\varphi/2 = 2k\pi \pm \pi/3, \quad 3\varphi/3 = 2k\pi \pm 2\pi/3;$$

consequently,

$$\varphi = \frac{4}{3} k\pi \pm \frac{2}{9} \pi, \quad \varphi = \frac{4}{3} k\pi \pm \frac{4}{9} \pi.$$

On the circle (ABC) there are altogether six points with such arguments and they are obtained by rotations of the radii OT_1, OT_3, OT_5 (T_1, T_3, T_5 are the Boutain points of $\triangle ABC$) through the angles $\pm 40°$ (rotations of the radii OT_1, OT_2, OT_3 through the angles $\pm 80°$ lead to the same six points).

8°. From the relation

$$z'' = \sigma_1 - (1 + p^{-3})z, \tag{84}$$

which define a similarity transformation that carries $\triangle ABC$ into $\triangle A''B''C''$, it follows that the affix ω' of the fixed point Ω' of the similarity transformation (84) is

$$\omega' = \frac{\sigma_1}{2 + p^{-3}}$$

[in relation (84), put $z'' = z$].

9°. From the relation obtained in item 8° it follows that if point P describes a unit circle, then the point Ω' describes the circle (Ω') obtained from the unit circle via the linear fractional transformation

$$z' = \frac{\sigma_1}{2 + z}.$$

We will now prove that (Ω') is an orthocentroidal circle of the triangle ABC, that is, a circle with diameter GH. The affix k of the midpoint K of segment GH is

$$k = \frac{1}{2}\left(\frac{\sigma_1}{3} + \sigma_1\right) = \frac{2}{3} \sigma_1.$$

Furthermore,

$$|z' - k|$$

$$= \left|\frac{\sigma_1}{2+z} - \frac{2}{3}\sigma_1\right| = \left|\frac{-\sigma_1 - 2\sigma_1 z}{3(2+z)}\right|$$

$$= \frac{|\sigma_1|}{3}\left|\frac{1+2z}{2+z}\right| = \frac{|\sigma_1|}{3}\frac{|z|\,|2+\bar{z}|}{|2+z|} = \frac{|\sigma_1|}{3},$$

hence,

$$K\Omega' = OH/3 = GH/2.$$

Consequently, the point Ω' describes a circle with diameter GH.

10°. The points Ω' and G coincide if and only if

$$\frac{\sigma_1}{2+p^{-3}} = \frac{\sigma_1}{3},$$

whence $p^3 = 1$, that is, the point P coincides with one of the points T_1, T_3, T_5.

11°. The points Ω' and H coincide if an only if

$$\frac{\sigma_1}{2+p^{-3}} = \sigma_1,$$

whence $p^3 = -1$, that is, the point P coincides with one of the points T_2, T_4, T_6.

12°. The centroid G'' of $\triangle\, A''B''C''$ corresponds to the centroid G of $\triangle\, ABC$ under the similarity transformation

$$z'' = \sigma_1 - (1+p^{-3})z.$$

That is, the affix g'' of point G'' is

$$g'' = \sigma_1 - (1+p^{-3})\frac{\sigma_1}{3} = \frac{\sigma_1}{3}(2-p^{-3}).$$

From this we have

$$|g'' - k| = \left|\frac{\sigma_1}{3}(2-p^{-3}) - \frac{2\sigma_1}{3}\right| = \frac{|\sigma_1|}{3}.$$

That is, if the point P describes the unit circle, then the point G'' describes the orthocentroidal circle (GH) of $\triangle\, ABC$.

13°. Eliminating p from the relations

$$\omega' = \frac{\sigma_1}{2+p^{-3}}, \quad g'' = \frac{\sigma_1}{3}(2-p^{-3}),$$

we obtain

$$2 + p^{-3} = \frac{\sigma_1}{\omega'}, \quad 2 - p^{-3} = \frac{3g''}{\sigma_1},$$

whence

$$3g''\,\omega' - 4\omega'\sigma_1 + \sigma_1^2 = 0.$$

In this relation, σ_1 is fixed and so it expresses g'' as a linear fractional function of ω' (both points G'' and Ω with affixes g'' and ω' lie on a circle with diameter GH) and therefore this function transforms the orthocentroidal circle (GH) of $\triangle\, ABC$ into itself. We find the fixed points of this transformation from the equation $3z^2 - 4\sigma_1 z + \sigma_1^2 = 0$, whence

$$z = \sigma_1, \quad z = \sigma_1/3$$

are the affixes of the points H and G.

14°. The affix o_1 of point O_1 is

$$o_1 = \frac{4}{3}\sigma_1.$$

The slope of line GH is equal to $\frac{\sigma_1}{\bar{\sigma}_1}$. Hence, the equation of the straight line passing through point Ω' parallel to GH is

$$z - \omega' = \frac{\sigma_1}{\bar{\sigma}_1}(\bar{z} - \bar{\omega}')$$

or

$$z - \frac{\sigma_1}{\bar{\sigma}_1}\bar{z} = \omega' - \frac{\sigma_1}{\bar{\sigma}_1}\bar{\omega}'. \tag{85}$$

The equation of the midperpendicular of line segment GH is

$$z - \frac{2}{3}\sigma_1 = -\frac{\sigma_1}{\sigma_1}\left(\bar{z} - \frac{2}{3}\bar{\sigma}_1\right)$$

or

$$z + \frac{\sigma_1}{\bar{\sigma}_1}\bar{z} = \frac{4}{3}\sigma_1. \tag{86}$$

Adding termwise the equations (85) and (86), we find the affix $z = \pi$ of the projection of point Ω' on the midperpendicular of segment GH:

$$\pi = \frac{1}{2}\left(\omega' + \frac{4}{3}\sigma_1 - \frac{\sigma_1}{\bar{\sigma}_1}\bar{\omega}'\right).$$

The affix ω^* of the point Ω^* can be found from the relation

$$\frac{\omega' + \omega^*}{2} = \pi = \frac{1}{2}\left(\omega' + \frac{4}{3}\sigma_1 - \frac{\sigma_1}{\bar{\sigma}_1}\bar{\omega}'\right),$$

whence

$$\omega^* = \frac{4}{3}\sigma_1 - \frac{\sigma_1}{\bar{\sigma}_1}\bar{\omega}'$$

but

$$\bar{\omega}' = \frac{\bar{\sigma}_1}{2 + p^{-3}}.$$

Hence,

$$\omega^* = \frac{4}{3}\sigma_1 - \frac{\sigma_1}{2 + \bar{p}^{-3}}.$$

We have

$$\begin{vmatrix} o_1 & \omega^* & g'' \\ \bar{o}_1 & \bar{\omega}^* & g'' \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \frac{4}{3}\sigma_1 & \frac{4}{3}\sigma_1 - \frac{\sigma_1}{2 + \bar{p}^{-3}} & \frac{\sigma_1}{3}(2 - p^{-3}) \\ \frac{4}{3}\bar{\sigma}_1 & \frac{4}{3}\bar{\sigma}_1 - \frac{\bar{\sigma}_1}{2 + p^{-3}} & \frac{\bar{\sigma}_1}{3}(2 - p^{-3}) \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \sigma_1\bar{\sigma}_1 \begin{vmatrix} \frac{4}{3} & -\frac{1}{2 + \bar{p}^{-3}} & -\frac{1}{3}(2 + p^{-3}) \\ \frac{4}{3} & -\frac{1}{2 + p^{-3}} & -\frac{1}{3}(2 + \bar{p}^{-3}) \\ 1 & 0 & 0 \end{vmatrix} = 0.$$

Problem 27. Let P, Q, R be the orthogonal projections of point M on the sides BC, CA, AB of $\triangle ABC$; let A_0, B_0, C_0 be the midpoints of segments MA, MB, MC, and let A', B', C' be the points obtained by inversion of the points A_0, B_0, C_0 with the circle of inversion (PQR). Prove that the ratio of the area of $\triangle \overrightarrow{A'B'C'}$ to the area of $\triangle \overrightarrow{PQR}$ is equal to the ratio of the square of the radius of (PQR) to the power of the point M with respect to that circle with sign reversed (Fig. 30).

Solution. Take (PQR) as the unit circle. Let z_1, z_2, z_3, z_0 be the respective affixes of the points P, Q, R, M. Since the radius of (PQR) is regarded as equal to 1, it follows that the power of the point M with respect to the

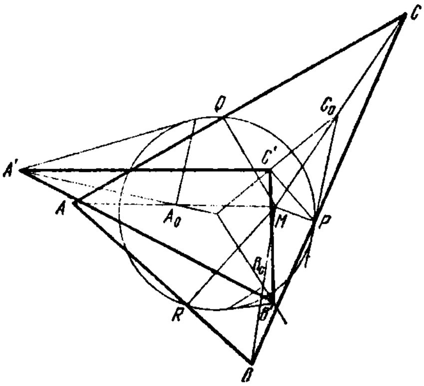

Fig. 30

circle (PQR) is equal to $\sigma = z_0\bar{z}_0 - 1$. We thus have to prove that

$$\frac{(A'B'C')}{(PQR)} = -\frac{1}{\sigma}.$$

The equation of the straight line MQ is of the form

$$z - z_0 = \frac{z_0 - z_2}{\bar{z}_0 - \bar{z}_2}(\bar{z} - \bar{z}_0)$$

or

$$(\bar{z}_0 - \bar{z}_2)z - (z_0 - z_2)\bar{z} = \bar{z}_0 z_2 - z_0\bar{z}_2.$$

The slope of the straight line AQ is equal to $-\dfrac{z_0 - z_2}{\bar{z}_0 - \bar{z}_2}$ and therefore the equation of this line is of the form

$$z - z_2 = -\frac{z_0 - z_2}{\bar{z}_0 - \bar{z}_2}(\bar{z} - \bar{z}_2)$$

or

$$(\bar{z}_0 - \bar{z}_2)z + (z_0 - z_2)\bar{z} = z_2\bar{z}_0 + \bar{z}_2 z_0 - 2. \tag{87}$$

Similarly, we can obtain the equation of the straight line AR:

$$(\bar{z}_0 - \bar{z}_3)z + (z_0 - z_3)\bar{z} = z_3\bar{z}_0 + \bar{z}_3 z_0 - 2. \tag{88}$$

Solving the system of equations (87) and (88), we find the affix $z = a$ of point A:

$$a = \frac{\Delta'}{\Delta} = \frac{\begin{vmatrix} z_2\bar{z}_0 + \bar{z}_2 z_0 - 2 & z_0 - z_2 \\ z_3\bar{z}_0 + \bar{z}_3 z_0 - 2 & z_0 - z_3 \end{vmatrix}}{\begin{vmatrix} \bar{z}_0 - \bar{z}_2 & z_0 - z_2 \\ \bar{z}_0 - \bar{z}_3 & z_0 - z_3 \end{vmatrix}}.$$

We have

$$\Delta' = (z_2\bar z_0 + \bar z_2 z_0 - 2)(z_0 - z_3) - (z_3\bar z_0 + \bar z_3 z_0 - 2)(z_0 - z_2)$$

$$= z_2\bar z_0 z_0 + \bar z_2 z_0^2 - 2z_0 - z_2 z_3\bar z_0 - z_3\bar z_2 z_0 + 2z_3$$

$$- z_3 z_0\bar z_0 - \bar z_3 z_0^2 + 2z_0 + z_2 z_3\bar z_0 + \bar z_3 z_2 z_0 - 2z_2$$

$$= (z_2 - z_3)z_0\bar z_0 + z_0^2\left(\frac{1}{z_2} - \frac{1}{z_3}\right) + z_0\left(\frac{z_2}{z_3} - \frac{z_3}{z_2}\right) - 2(z_2 - z_3)$$

$$= (z_2 - z_3)\left(z_0\bar z_0 - \frac{z_0^2}{z_2 z_3} + \frac{z_2 + z_3}{z_2 z_3} z_0 - 2\right)$$

$$= (z_2 - z_3)\frac{z_0\bar z_0 z_2 z_3 - z_0^2 + (z_2 + z_3)z_0 - 2z_2 z_3}{z_2 z_3},$$

$$\Delta = (\bar z_0 - \bar z_2)(z_0 - z_3) - (z_0 - z_2)(\bar z_0 - \bar z_3)$$

$$= \bar z_0 z_0 - \bar z_0 z_3 - \bar z_2 z_0 + \bar z_2 z_3 - z_0\bar z_0 + z_0\bar z_3 + z_2\bar z_0 - z_2\bar z_3$$

$$= \bar z_0(z_2 - z_3) + z_0\left(\frac{1}{z_3} - \frac{1}{z_2}\right) + \frac{z_3}{z_2} - \frac{z_2}{z_3}$$

$$= (z_2 - z_3)\frac{z_0 + z_2 z_3\bar z_0 - z_2 - z_3}{z_2 z_3}.$$

Thus

$$a = \frac{-z_0^2 + (z_2 z_3\bar z_0 + z_2 + z_3)z_0 - 2z_2 z_3}{z_0 + z_2 z_3\bar z_0 - z_2 - z_3}.$$

The affix a_0 of the midpoint A_0 of segment AM is

$$a_0 = \frac{z_0 + a}{2} = \frac{z_0^2 + z_2 z_3 z_0\bar z_0 - z_2 z_0 - z_3 z_0 - z_0^2 + z_2 z_3 z_0\bar z_0 + z_0 z_2 + z_0 z_3 - 2z_2 z_3}{2(z_0 + z_2 z_3\bar z_0 - z_2 - z_3)}$$

$$= \frac{z_2 z_3(z_0\bar z_0 - 1)}{z_0 + z_2 z_3\bar z_0 - z_2 - z_3}.$$

The affix a' of the point A' is equal to $1/\bar a_0$, that is,

$$a' = \frac{\bar z_0 + \dfrac{1}{z_2 z_3} z_0 - \dfrac{1}{z_2} - \dfrac{1}{z_3}}{\dfrac{1}{z_2 z_3}(z_0\bar z_0 - 1)} = \frac{z_0 + z_2 z_3\bar z_0 - z_2 - z_3}{z_0\bar z - 1}$$

$$= \frac{1}{\sigma}(z_0 + \bar z_0\bar z_1\sigma_3 + z_1 - \sigma_1).$$

We now have

$$\bar{a}' = \frac{1}{\sigma}(\bar{z}_0 + z_0 z_1 \bar{\sigma}_3 + \bar{z}_1 - \bar{\sigma}_1).$$

Similarly,

$$b' = \frac{1}{\sigma}(z_0 + \bar{z}_0 \bar{z}_2 \sigma_3 + z_2 - \sigma_1),$$

$$c' = \frac{1}{\sigma}(z_0 + \bar{z}_0 \bar{z}_3 \sigma_3 + z_3 - \sigma_1),$$

$$\bar{b}' = \frac{1}{\sigma}(\bar{z}_0 + z_0 z_2 \bar{\sigma}_3 + \bar{z}_2 - \bar{\sigma}_1),$$

$$\bar{c}' = \frac{1}{\sigma}(\bar{z}_0 + z_0 z_3 \bar{\sigma}_3 + \bar{z}_3 - \bar{\sigma}_1)$$

and, hence,

$$(A'B'C') = \frac{i}{4}\begin{vmatrix} a' & \bar{a}' & 1 \\ b' & \bar{b}' & 1 \\ c' & \bar{c}' & 1 \end{vmatrix} = \frac{i}{4\sigma^2}\begin{vmatrix} z_0+\bar{z}_0\bar{z}_1\sigma_3+z_1-\sigma_1 & \bar{z}_0+z_0z_1\bar{\sigma}_3+\bar{z}_1-\bar{\sigma}_1 & 1 \\ z_0+\bar{z}_0\bar{z}_2\sigma_3+z_2-\sigma_1 & \bar{z}_0+z_0z_2\bar{\sigma}_3+\bar{z}_2-\bar{\sigma}_1 & 1 \\ z_0+\bar{z}_0\bar{z}_3\sigma_3+z_3-\sigma_1 & \bar{z}_0+z_0z_3\bar{\sigma}_3+\bar{z}_3-\bar{\sigma}_1 & 1 \end{vmatrix}$$

$$= \frac{i}{4\sigma^2}\begin{vmatrix} \bar{z}_0\bar{z}_1\sigma_3+z_1 & z_0z_1\bar{\sigma}_3+\bar{z}_1 & 1 \\ \bar{z}_0\bar{z}_2\sigma_3+z_2 & z_0z_2\bar{\sigma}_3+\bar{z}_2 & 1 \\ \bar{z}_0\bar{z}_3\sigma_3+z_3 & z_0z_3\bar{\sigma}_3+\bar{z}_3 & 1 \end{vmatrix} = \frac{i}{4\sigma^2}\left[z_0\bar{z}_0\begin{vmatrix} \bar{z}_1 & z_1 & 1 \\ \bar{z}_2 & z_2 & 1 \\ \bar{z}_3 & z_3 & 1 \end{vmatrix} + \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}\right]$$

$$= -\frac{1}{\sigma}(PQR)$$

whence

$$\frac{(A'B'C')}{(PQR)} = -\frac{1}{\sigma}.$$

Problem 28. $A_1A_2A_3$ is an arbitrary triangle; P is an arbitrary point. Through P draw two mutually perpendicular straight lines δ and δ'. Let these lines intersect side A_2A_3 at the points B_2 and B_3, respectively, and the altitude (dropped from point A_1 on side A_2A_3) at points B_2' and B_3'. In similar fashion construct the points C_3 and C_1, C_3' and C_1' for side A_3A_1 and the altitude dropped from vertex A_2 on that side; also construct the points D_1 and D_2, D_1' and D_2' for side A_1A_2 and the altitude dropped from point A_3 on side A_1A_2. Prove that the centroids of the three groups of points B_2, B_3, B_2', B_3'; C_3, C_1, C_3', C_1'; D_1, D_2, D_1', D_2' lie on a single straight line (to all indicated twelve points are assigned identical masses, see Fig. 31).

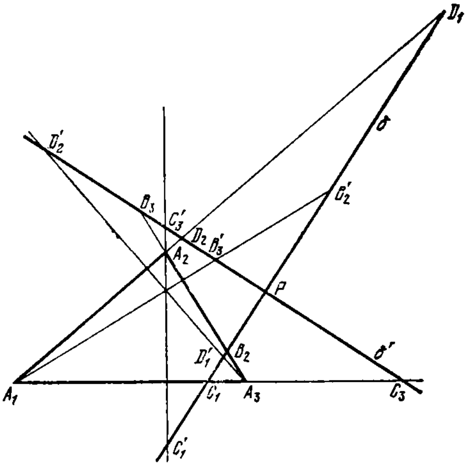

Fig. 31

Solution. Take $(A_1A_2A_3)$ for the unit circle. Let z_1, z_2, z_3, p be the respective affixes of the points A_1, A_2, A_3, P. Let us write down the equations of the straight lines δ and δ' in the form

$$z - p = \tau(\bar{z} - \bar{p}) \text{ or } z - \tau\bar{z} = p - \tau\bar{p}, \tag{89}$$

$$z - p = -\tau(\bar{z} - \bar{p}) \text{ or } z + \tau\bar{z} = p + \tau\bar{p}, \tag{90}$$

where $|\tau| = 1$. The equation of the straight line A_2A_3 is of the form

$$z + z_2z_3\bar{z} = z_2 + z_3. \tag{91}$$

Subtracting (89) from (91) term by term, we obtain the number $\bar{z} = \bar{b}_2$, which is the conjugate of the affix b_2 of point B_2:

$$\bar{b}_2 = \frac{z_2 + z_3 - p + \tau\bar{p}}{z_2 z_3 + \tau}, \tag{92}$$

whence

$$b_2 = \frac{\dfrac{1}{z_2} + \dfrac{1}{z_3} - \bar{p} + \dfrac{1}{\tau}p}{\dfrac{1}{z_2 z_3} + \dfrac{1}{\tau}} = \frac{\tau(z_2 + z_3 - z_2 z_3 \bar{p}) + z_2 z_3 p}{z_2 z_3 + \tau}. \tag{93}$$

Substituting $-\tau$ for τ in the formulas just obtained, we find the affix b_3 of point B_3 and $\bar{b}_3$:

$$b_3 = \frac{-\tau(z_2 + z_3 - z_2 z_3 \bar{p}) + z_2 z_3 p}{z_2 z_3 - \tau},$$

$$\bar{b}_3 = \frac{z_2 + z_3 - p - \tau\bar{p}}{z_2 z_3 - \tau}.$$

From these formulas we find the number $\bar{b}_{23}$, which is the conjugate of the affix b_{23} of the midpoint B_{23} of segment B_2B_3:

$$\bar{b}_{23} = \frac{\bar{b}_2 + \bar{b}_3}{2} = \frac{1}{2}\left(\frac{z_2 + z_3 - p + \tau\bar{p}}{z_2 z_3 + \tau} + \frac{z_2 + z_3 - p - \tau p}{z_2 z_3 - \tau}\right)$$

$$= \frac{z_2 z_3(z_2 + z_3) - p z_2 z_3 - \tau^2 \bar{p}}{z_2^2 z_3^2 - \tau^2},$$

whence

$$b_{23} = \frac{\dfrac{1}{z_2 z_3}\left(\dfrac{1}{z_2} + \dfrac{1}{z_3}\right) - \bar{p}\dfrac{1}{z_2 z_3} - \dfrac{1}{\tau^2} p}{\dfrac{1}{z_2^2 z_3^2} - \dfrac{1}{\tau^2}} = \frac{z_2^2 z_3^2 p + \tau^2 \bar{p} z_2 z_3 - (z_2 + z_3)\tau^2}{z_2^2 z_3^2 - \tau^2}.$$

Furthermore, the equation of the straight line passing through point A_1 perpendicularly to line A_2A_3 is of the form

$$z - z_1 = z_2 z_3(\bar{z} - \bar{z}_1)$$

or

$$z - z_2 z_3 \bar{z} = z_1 - \frac{z_2 z_3}{z_1}. \tag{94}$$

Subtracting (94) from (89) term by term, we find the affix $\bar{z} = \bar{b}_2'$, which is the conjugate of affix b_2' of point B_2':

$$\bar{b}_2' = \frac{p - \tau\bar{p} - z_1 + \dfrac{z_2 z_3}{z_1}}{z_2 z_3 - \tau}$$

and, similarly,

$$\bar{b}_3' = \frac{p + \tau\bar{p} - z_1 + \dfrac{z_2 z_3}{z_1}}{z_2 z_3 + \tau}.$$

The number $\bar{b}_{23}'$, which is the conjugate of the affix b_{23}' of the midpoint B_{23}' of segment $B_2'B_3'$, is

$$\bar{b}_{23}' = \frac{1}{2}\left(\frac{p - \tau\bar{p} - z_1 + \dfrac{z_2 z_3}{z_1}}{z_2 z_3 - \tau} + \frac{p + \tau\bar{p} - z_1 + \dfrac{z_2 z_3}{z_1}}{z_2 z_3 + \tau}\right)$$

$$= \frac{z_2 z_3 p - \sigma_3 + \dfrac{z_2^2 z_3^2}{z_1} - \tau^2 \bar{p}}{z_2^2 z_3^2 - \tau^2},$$

whence

$$b'_{23} = \frac{\dfrac{1}{z_2 z_3}\bar{p} - \dfrac{1}{\sigma_3} + \dfrac{z_1}{z_2^2 z_3^2} - \dfrac{1}{\tau^2}p}{\dfrac{1}{z_2^2 z_3^2} - \dfrac{1}{\tau^2}} = \frac{pz_2^2 z_3^2 - z_1 \tau^2 + \dfrac{z_2 z_3}{z_1}\tau^2 - z_2 z_3 \bar{p}\tau^2}{z_2^2 z_3^2 - \tau^2}.$$

The affix λ of the centroid of the set of points B_2, B_3, B'_2, B'_3 is equal to the affix of the midpoint of segment $B_{23}B'_{23}$, the ends B_{23} and B'_{23} of which are the midpoints of segments B_2B_3 and $B'_2B'_3$:

$$\lambda = \frac{b_{23} + b'_{23}}{2} = \frac{2z_2^2 z_3^2 p - \tau^2\left(\sigma_1 - \dfrac{z_2 z_3}{z_1}\right)}{2(z_2^2 z_3^2 - \tau^2)}. \tag{95}$$

Let us now prove that the point with affix λ lies on the straight line Δ given by the equation *

$$2z\left(2\bar{p} + \frac{\sigma_3}{\tau^2} - \bar{\sigma}_1\right) + 2\bar{z}\left(2p + \frac{\tau^2}{\sigma_3} - \sigma_1\right)$$
$$= \left(2p + \frac{\tau^2}{\sigma_3}\right)\left(2\bar{p} + \frac{\sigma_3}{\tau^2}\right) - \sigma_1\bar{\sigma}_1. \tag{96}$$

We have

$$\bar{\lambda} = \frac{2\bar{p}\dfrac{1}{z_2^2 z_3^2} - \dfrac{1}{\tau^2}\left(\bar{\sigma}_1 - \dfrac{z_1}{z_2 z_3}\right)}{2\left(\dfrac{1}{z_2^2 z_3^2} - \dfrac{1}{\tau^2}\right)} = \frac{\bar{\sigma}_1 z_2^2 z_3^2 - \sigma_3 - 2\bar{p}\tau^2}{2(z_2^2 z_3^2 - \tau^2)}. \tag{97}$$

---

* The equation of this straight line could have been set up by similarly determining the affix μ of the centroid of the second set of points C_3, C_1, C'_3, C'_1:

$$\mu = \frac{2z_3^2 z_1^2 p - \tau^2\left(\sigma_1 - \dfrac{z_3 z_1}{z_2}\right)}{2(z_3^2 z_1^2 - \tau^2)};$$

also, by determining the affix ν of the set of points D_1, D_2, D'_1, D'_2, it is possible to verify the validity of the necessary and sufficient condition of collinearity of the three points

$$\begin{vmatrix} \lambda & \bar{\lambda} & 1 \\ \mu & \bar{\mu} & 1 \\ \nu & \bar{\nu} & 1 \end{vmatrix} = 0.$$

Let us compute the left-hand side of (96), setting $z = \lambda$, $\bar{z} = \bar{\lambda}$ (the denominator $z_2^2 z_3^2 - \tau^2$ is dropped for the time being):

$$\left(2p\, z_2^2 z_3^2 - \tau^2 \sigma_1 + \tau^2 \frac{z_2 z_3}{z_1}\right)\left(2\bar{p} + \frac{\sigma_3}{\tau^2} - \frac{\sigma_2}{\sigma_3}\right)$$

$$+ \left(z_2^2 z_3^2 \frac{\sigma_2}{\sigma^3} - \sigma_3 - 2\bar{p}\, \tau^2\right)\left(2p + \frac{\tau^2}{\sigma^3} - \sigma_1\right)$$

$$= 4p\,\bar{p} z_2^2 z_3^2 + 2p\, z_2^2 z_3^2 \frac{\sigma_3}{\tau^2} - 2p \frac{\sigma_2}{\sigma_3} z_2^2 z_3^2 - 2\bar{p}\, \tau^2 \sigma_1 - \sigma_1 \sigma_3 + \frac{\sigma_1 \sigma_2}{\sigma_3} \tau^2$$

$$+ 2\bar{p}\, \tau^2 \frac{z_2 z_3}{z_1} + \sigma_3 \frac{z_2 z_3}{z_1} - \frac{\sigma_2}{\sigma_3} \tau^2 \frac{z_2 z_3}{z_1} + 2p\, z_2^2 z_3^2 \frac{\sigma_2}{\sigma_3} + \tau^2 \frac{\sigma_2}{\sigma_3^2} z_2^2 z_3^2$$

$$- \frac{\sigma_1 \sigma_2}{\sigma_3} z_2^2 z_3^2 - 2p\, \sigma_3 - \tau^2 + \sigma_3 \sigma_1 - 4p\, \bar{p}\tau^2 - 2\bar{p} \frac{\tau^4}{\sigma_3} + 2\bar{p}\, \tau^2 \sigma_1$$

$$= 4p\, \bar{p}(z_2^2 z_3^2 - \tau^2) + z_2^2 z_3^2 - \tau^2 - \frac{\sigma_1 \sigma_2}{\sigma_3} (z_2^2 z_3^2 - \tau^2)$$

$$+ 2p \frac{\sigma_3}{\tau^2} (z_2^2 z_3^2 - \tau^2) + \frac{2\bar{p}\, \tau^2}{\sigma_3} (z_2^2 z_3^2 - \tau^2).$$

Thus, the left-hand side is, for $z = \lambda$, equal to

$$4p\bar{p} + 1 - \frac{\sigma_1 \sigma_2}{\sigma_3} + 2p \frac{\sigma_3}{\tau^2} + \frac{2\bar{p}\, \tau^2}{\sigma_3}.$$

The right-hand side of equation (96) is

$$4p\bar{p} + \frac{2p\, \sigma_3}{\tau^2} + 2\bar{p} \frac{\tau^2}{\sigma_3} + 1 - \frac{\sigma_1 \sigma_2}{\sigma_3},$$

which is the same as the left-hand side. The symmetry of equation (96) proves the theorem. At the same time, (96) is the equation of straight line on which lie three centroids of three sets of points:

$$B_2, B_3, B_2', B_3';\ \ C_3, C_1, C_3', C_1';\ \ D_1, D_2, D_1', D_2'.$$

From this theorem there follows a corollary (the *Droz-Farny theorem*) (see Fig. 32): if through the orthocentre H of triangle $A_1A_2A_3$ we draw two mutually perpendicular straight lines, which cut off on sides A_2A_3, A_3A_1, A_1A_2 segments B_2B_3, C_3C_1, D_1D_2, then the midpoints of segments B_2B_3, C_3C_1, D_1D_2 lie on one straight line. Of course, this theorem can be proved in a straightforward fashion, and its proof is technically not so elegant as that given in the problem generalizing the Droz-Farny theorem. In a direct proof of the Droz-Farny theorem, one should bear in mind that σ_1 is the affix of the orthocentre H of the triangle $A_1A_2A_3$.

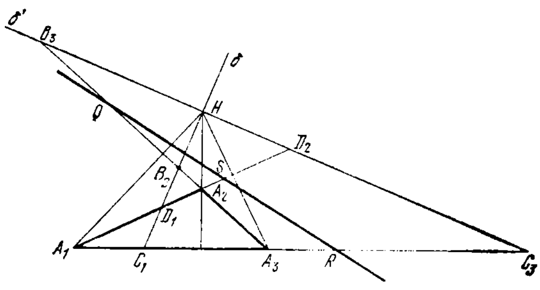

Fig. 32

It is obvious that the Droz-Farny line is obtained from the straight line Δ given in the generalization (for the same case $P = H$) under the homothetic transformation $\left(H, \frac{1}{2}\right)$; the points B_2', B_3' merge with the point H, and the centroid of the set of points B_2, B_3, B_2', B_3' is the midpoint of segment HB_{23}, where B_{23} is the midpoint of segment B_2B_3. And similarly for the points C_3', C_1' and D_1', D_2'.

Problem 29. Let I be the centre of a circle (I) inscribed in $\triangle ABC$; let D, E, F be points of tangency of (I) with the sides BC, CA, AB, respectively; let H be the orthocentre of $\triangle DEF$; let δ be some diameter of (I); let D', E', F' be points symmetric to the corresponding points D, E, F with respect to the straight line δ. Prove that if P_a, P_b, P_c are points symmetric to an arbitrary point P with respect to the sides $E'F', F'D', D'E'$ of $\triangle D'E'F'$ and α, β, γ are points symmetric to the points P_a, P_b, P_c respectively with respect to the sides BC, CA, AB of $\triangle ABC$, then the centre Q of the circle $(\alpha\beta\gamma)$ has the property that the Simson line of point Q with respect to $\triangle DEF$ is parallel to the straight line δ. Prove that if we construct a directed line segment $\overrightarrow{QN}$ equivalent to a directed line segment $\overrightarrow{HP^*}/2$, where P^* is a point symmetric to point P with respect to the straight line δ, then the radius of $(\alpha\beta\gamma)$ will be equal to $2IN$.

Solution. Take $(I) = (DEF)$ for the unit circle and take the diameter δ for the real axis. Let z_1, z_2, z_3, p be the affixes of the points D, E, F, P. Then the affixes b and c of points B and C are

$$b = \frac{2z_3 z_1}{z_3 + z_1}, \quad c = \frac{2z_1 z_2}{z_1 + z_2}$$

and their conjugates are

$$\overline{b} = \frac{2}{z_3 + z_1}, \quad \overline{c} = \frac{2}{z_1 + z_2}.$$

Let us find the affix p_a of point P_a. The affixes of the points D', E', F' are $\bar{z}_1, \bar{z}_2, \bar{z}_3$. The slope of the straight line $E'F'$ is

$$\frac{\bar{z}_2-\bar{z}_3}{z_2-z_3}=\frac{\dfrac{1}{z_2}-\dfrac{1}{z_3}}{z_2-z_3}=-\frac{1}{z_2 z_3}.$$

The equation of the straight line $E'F'$ is

$$z-\bar{z}_2=\frac{1}{z_2 z_3}(\bar{z}-z_2)$$

or

$$z+\bar{z}_2\bar{z}_3\bar{z}=\bar{z}_2+\bar{z}_3. \tag{98}$$

The equation of the perpendicular dropped from point P to this line is

$$z-p=\frac{1}{z_2 z_3}(\bar{z}-\bar{p})$$

or

$$z-\bar{z}_2\bar{z}_3\bar{z}=p-\bar{z}_2\bar{z}_3\bar{p}. \tag{99}$$

From equations (98) and (99) we find the affix p_a^* of the projection of point P on the straight line $E'F'$:

$$p_a^*=\frac{1}{2}(\bar{z}_2+\bar{z}_3+p-\bar{z}_2\bar{z}_3\bar{p}).$$

The affix p_a of point P_a, which is symmetric to point P about the straight line $E'F'$, can be found from the relation

$$\frac{p+p_a}{2}=p_a^*=\frac{1}{2}(\bar{z}_2+\bar{z}_3-\bar{z}_2\bar{z}_3\bar{p}+p),$$

whence

$$p_a=\bar{z}_2+\bar{z}_3-\bar{z}_2\bar{z}_3\bar{p}.$$

The slope of the straight line ID is $z_1/\bar{z}_1=z_1^2$, and the slope of BC is equal to $-z_1^2$. The equation of BC is

$$z-z_1=-z_1^2(\bar{z}-\bar{z}_1)$$

or

$$z+z_1^2\bar{z}=2z_1. \tag{100}$$

The equation of the perpendicular dropped from point P_a on the side BC is

$$z-p_a=z_1^2(\bar{z}-\bar{p}_a)$$

or

$$z - z_1^2 \bar{z} = p_a - z_1^2 \bar{p}_a. \tag{101}$$

From equations (100) and (101) we find the affix $\tilde{p}_a$ of the projection of point P_a on line BC:

$$\tilde{p}_a = \frac{1}{2}(p_a - z_1^2 \bar{p}_a + 2z_1)$$

and from the relation

$$\frac{\lambda + p_a}{2} = \tilde{p}_a = \frac{1}{2}(p_a - z_1^2 \bar{p}_a + 2z_1)$$

we find the affix λ of point α:

$$\begin{aligned}\lambda = 2z_1 - z_1^2(z_2 + z_3 - z_2 z_3 p) &= 2z_1 - z_1(z_1 z_2 + z_1 z_3 - \sigma_3 p)\\ &= 2z_1 - z_1(\sigma_2 - z_2 z_3 - \sigma_3 p) = \sigma_3 + z_1(2 - \sigma_2 + p\sigma_3).\end{aligned}$$

The affixes μ and ν of points β and γ have similar expressions:

$$\mu = \sigma_3 + z_2(2 - \sigma_2 + p\sigma_3),$$
$$\nu = \sigma_3 + z_3(2 - \sigma_2 + p\sigma_3).$$

Thus we see that the points α, β, γ lie on a circle, the affix of whose centre is equal to σ_3, and since $|\sigma_3| = 1$, it follows that this centre lies on the circle (DEF). The radius of the circle $(\alpha\beta\gamma)$ is

$$\begin{aligned}\rho = |2 - \sigma_2 + p\sigma_3| = |2 - \bar{\sigma}_2 + \bar{p}\bar{\sigma}_3| &= \left|2 - \frac{\sigma_1}{\sigma_3} + \frac{\bar{p}}{\sigma_3}\right|\\ = \frac{|2\sigma_3 - \sigma_1 + \bar{p}|}{|\sigma_3|} = |2\sigma_3 - \sigma_1 + \bar{p}| &= 2\left|\sigma_3 + \frac{1}{2}(\bar{p} - \sigma_1)\right|.\end{aligned}$$

The fact that the Simson line corresponding to the point with affix σ_3 is parallel to the real axis follows from problem 3.

The affix p^* of point P^*, which is symmetric to point P about the straight line δ, is equal to $p^* = \bar{p}$. To the directed line segment $\overrightarrow{HP^*}$ there corresponds a complex number $p^* - \sigma_1 = \bar{p} - \sigma_1$.*

The directed line segment $\overrightarrow{HP^*}/2 \equiv \overrightarrow{QN}$ is associated with the complex number $(\bar{p} - \sigma_1)/2 = n - \sigma_3$, where n is the affix of point N. From this we have $n = \sigma_3 + (\bar{p} - \sigma_1)/2$. Since n is the affix of point N and (I) is taken as the unit circle, it follows that

$$IN = |n| = \left|\sigma_3 + \frac{1}{2}(\bar{p} - \sigma_1)\right| = \frac{1}{2}\rho.$$

* Let the points A and B have affixes a and b, respectively. We will say that the directed line segment $\overrightarrow{AB}$ is associated with the complex number $b - a$.

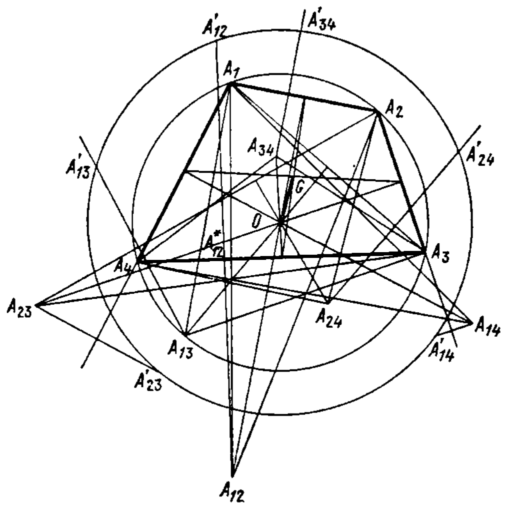

Fig. 33

Problem 30. On the circle (O) take four arbitrary points A_1, A_2, A_3, A_4. Take any one of six segments joining these points in pairs, for example, segment A_1A_2. Construct an isosceles triangle $A_1A_2A_{12}$ with vertex A_{12} and base A_1A_2 so that its centroid coincides with the point O. Let A'_{12} be a point symmetric to point A_{12} about the straight line A_3A_4. Prove that the six points thus obtained, $A'_{12}, A'_{13}, A'_{14}, A'_{23}, A'_{24}, A'_{34}$, lie on one circle, (S), that is concentric with (O); note that the radius of (S) is four times the distance from point O to the centroid G of the system of four points A_1, A_2, A_3, A_4 (Fig. 33).

Solution. Take (O) for the unit circle and let z_1, z_2, z_3, z_4 be the respective affixes of points A_1, A_2, A_3, A_4. The affix α_{12} of the midpoint of segment A_1A_2 is $\alpha_{12} = (z_1 + z_2)/2$; hence, the affix a_{12} of vertex A_{12} of the isosceles triangle $A_1A_2A_{12}$ ($A_1A_{12} = A_2A_{12}$), the centroid of which coincides with O, is

$$a_{12} = -(z_1 + z_2).$$

The equation of the straight line A_3A_4 is

$$z + z_3 z_4 \bar{z} = z_3 + z_4. \tag{102}$$

The equation of the perpendicular dropped from the point A_{12} to the straight line A_3A_4 is

$$z + z_1 + z_2 = z_3 z_4(\bar{z} + \bar{z}_1 + \bar{z}_2)$$

or

$$z - z_3 z_4 \bar{z} = -z_1 - z_2 + z_3 z_4 \bar{z}_1 + z_3 z_4 \bar{z}_2. \tag{103}$$

Adding equations (102) and (103) termwise, we find the affix $z = a_{12}^*$ of the projection of point A_{12} on A_3A_4:

$$a_{12}^* = \frac{1}{2}(z_3 + z_4 - z_1 - z_2 + z_3 z_4 \overline{z}_1 + \overline{z}_3 z_4 \overline{z}_2).$$

The affix a'_{12} of point A'_{12} is found from the relation

$$\frac{a_{12} + a'_{12}}{2} = a_{12}^*$$

or

$$\frac{-z_1 - z_2 + a'_{12}}{2} = \frac{1}{2}(z_3 + z_4 - z_1 - z_2 + z_3 z_4 \overline{z}_1 + z_3 z_4 \overline{z}_2),$$

whence

$$a'_{12} = z_3 + z_4 + z_3 z_4 \overline{z}_1 + z_3 z_4 \overline{z}_2 = z_3 z_4(\overline{z}_1 + \overline{z}_2 + \overline{z}_3 + \overline{z}_4) = z_3 z_4 \bar{\sigma}_1,$$

where $\sigma_1 = z_1 + z_2 + z_3 + z_4$. From this relation it follows that

$$OA'_{12} = |a'_{12}| = |\sigma_1| \quad (|z_3| = |z_4| = 1, \quad |\bar{\sigma}_1| = |\sigma_1|).$$

Similarly,

$$OA'_{13} = OA'_{14} = OA'_{23} = OA'_{24} = OA'_{34} = |\sigma_1|.$$

Thus, the points $A'_{12}, A'_{13}, A'_{14}, A'_{23}, A'_{24}, A'_{34}$ lie on one circle, whose radius is equal to $\rho = |\sigma_1| = 4|\sigma_1/4|$. But $\sigma_1/4$ is the affix of the centroid of the system of four points A_1, A_2, A_3, A_4; hence, $|\sigma_1/4| = OG$. Thus,

$$\rho = 4OG.$$

Problem 31. Let D, E, F be points of tangency of the sides BC, CA, AB of $\triangle ABC$ with an inscribed ciicle. Consider an arbitrary diameter δ of the circle $(I) = (DEF)$. Draw through points D^*, E^*, F^*, in which the altitudes of $\triangle DEF$ intersect the circle (DEF), straight lines parallel to the straight line δ, and let A', B', C' be points at which these lines intersect the circle (I) a second time. Denote by A^*, B^*, C^* points symmetric to the points A', B', C', respectively, with respect to the sides BC, CA, AB of $\triangle ABC$. Prove that the centroids G and G^* of $\triangle DEF$ and $\triangle A^*B^*C^*$ are symmetric about the orthopole ω of the straight line δ with respect to $\triangle DEF$ (Fig. 34).

Solution. Take (DEF) for the unit circle, and let the diameter δ be the real axis Ox (point O coincides with point I). Denote by z_1, z_2, z_3 the respective affixes of the points D, E, F. The equation of the straight line passing through point D perpendicularly to line EF is

$$z - z_1 = z_2 z_3(\overline{z} - \overline{z}_1).$$

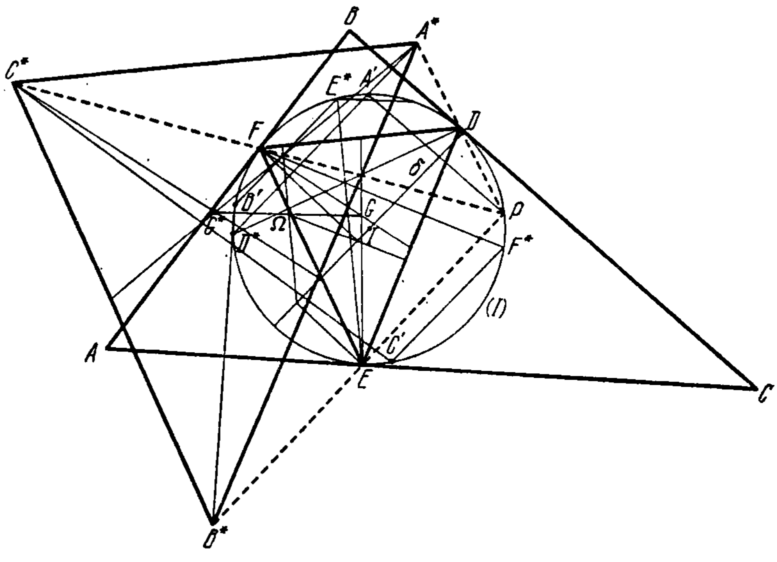

Fig. 34

Solving this equation together with the equation of the unit circle $z\bar{z} = 1$, we find the affix $z = d^*$ of the point D^*. Indeed,

$$z - z_1 = z_2 z_3\left(\frac{1}{z} - \frac{1}{z_1}\right)$$

or

$$z - z_1 = -\frac{z_2 z_3}{z_1 z}(z - z_1).$$

Naturally, one of the roots of this equation is $z = z_1$ (the affix of point D), the other root is

$$d^* = -\frac{z_2 z_3}{z_1},$$

which is the affix of point D^*. Similarly we find the affixes

$$e^* = -\frac{z_3 z_1}{z_2},\ f^* = -\frac{z_1 z_2}{z_3}$$

of points E^* and F^*.

The equation of the straight line passing, for example, through the point D^* parallel to line δ is of the form

$$z + \frac{z_2 z_3}{z_1} = \bar{z} + \frac{z_1}{z_2 z_3}.$$

Solving this equation jointly with the equation of the unit circle $z\bar{z} = 1$, we find the affix of point A':

$$z + \frac{z_2 z_3}{z_1} = \frac{1}{z} + \frac{z_1}{z_2 z_3},$$

$$\frac{z_1 z + z_2 z_3}{z_1} = \frac{z_1 z + z_2 z_3}{z_2 z_3 z}.$$

One of the roots of this equation, $z = -z_2 z_3/z_1$, is the affix d^* of point D^*, the other root is equal to the affix a' of point A':

$$a' = \frac{z_1}{z_2 z_3}.$$

Similarly,

$$b' = \frac{z_2}{z_3 z_1}, \quad c' = \frac{z_3}{z_1 z_2},$$

where b' and c' are the affixes of points B' and C' respectively.

The slope of ID is equal to $z_1/\bar{z}_1 = z_1^2$; hence, the equation of BC (the slope of which is $-z_1^2$) is

$$z - z_1 = -z_1^2(\bar{z} - \bar{z}_1)$$

or

$$z + z_1^2 \bar{z} = 2z_1. \tag{104}$$

The equation of the perpendicular dropped from point A' to line BC is

$$z - \frac{z_1}{z_2 z_3} = z_1^2 \left(\bar{z} - \frac{z_2 z_3}{z_1}\right)$$

or

$$z - z_1^2 \bar{z} = \frac{z_1}{z_2 z_3} - \sigma_3. \tag{105}$$

From (104) and (105) we find the affix α of the projection of point A' on line BC:

$$\alpha = \frac{1}{2}\left(2z_1 + \frac{z_1}{z_2 z_3} - \sigma_3\right).$$

The affix a^* of point A^* is found from the relation

$$\frac{a' + a^*}{2} = \alpha$$

or

$$\frac{\dfrac{z_1}{z_2 z_3} + a^*}{2} = \frac{1}{2}\left(2z_1 + \frac{z_1}{z_2 z_3} - \sigma_3\right),$$

whence

$$a^* = 2z_1 - \sigma_3$$

and, similarly,

$$b^* = 2z_2 - \sigma_3, \quad c^* = 2z_3 - \sigma_3,$$

where b^* and c^* are the affixes of the points B^* and C^* respectively.

From the relations obtained it follows that

$$\frac{a^* - \sigma_3}{z_1 - \sigma_3} = \frac{2z_1 - 2\sigma_3}{z_1 - \sigma_3} = 2$$

and, similarly, that

$$\frac{b^* - \sigma_3}{z_2 - \sigma_3} = 2, \quad \frac{c^* - \sigma_3}{z_3 - \sigma_3} = 2.$$

That is, the point P with affix $p = \sigma_3$ is the centre of the homothetic transformation under which $\triangle DEF$ goes into $\triangle A^*B^*C^*$ (see Fig. 34).

Furthermore, the affixes g and g^* of points G and G^* respectively are

$$g = \frac{\sigma_1}{3}, \quad g^* = \frac{2}{3}\sigma_1 - \sigma_3.$$

The midpoint Ω of segment GG^* has the affix $\omega = (\sigma_1 - \sigma_3)/2$ and, hence, coincides with the orthopole of the straight line δ (the real axis) [formula (11), problem 7, $z_0 = 0$, $\varkappa = 1$].

Remark. Let us find the affix of the point symmetric to point A' about DI: the equation of the perpendicular dropped from point A' to DI is of the form

$$z - \frac{z_1}{z_2 z_3} = -z_1^2\left(\bar{z} - \frac{z_2 z_3}{z_1}\right)$$

or

$$z + z_1^2 \bar{z} = \frac{z_1}{z_2 z_3} + \sigma_3.$$

The equation of DI is

$$z - z_1^2 \bar{z} = 0.$$

From this we find the affix $z = \lambda$ of the projection of point A' on line DI:

$$\lambda = \frac{1}{2}\left(\frac{z_1}{z_2 z_3} + \sigma_3\right).$$

The affix μ of the point symmetric to point A' about DI is found from the relation

$$\frac{a' + \mu}{2} = \lambda$$

or

$$\frac{\dfrac{z_1}{z_2 z_3} + \mu}{2} = \frac{1}{2}\left(\frac{z_1}{z_2 z_3} + \sigma_3\right),$$

whence

$$\mu = \sigma_3 = p.$$

Thus, all points symmetric to points A', B', C' about the straight lines DI, EI, FI coincide with the same point P, which is the centre of the homothetic transformation $\Gamma = (P, 2)$ that carries $\triangle DEF$ into $\triangle A^*B^*C^*$.

Problem 32. The circle $(DEF) = (I)$ inscribed in $\triangle ABC$ is taken as the unit circle; D, E, F are the respective points of tangency of the straight lines BC, CA, AB with the circle; z_1, z_2, z_3 are the respective affixes of the points D, E, F. It is required to find:

1°. The affix o of centre O of the circle $(O) = (ABC)$.

2°. The radius R of the circle (ABC).

3°. The radius ρ of the Euler circle of $\triangle ABC$.

4°. The affix h of the orthocentre H of $\triangle ABC$.

5°. The affix ε of the centre O_9 of the Euler circle (O_9) of $\triangle ABC$.

6°. Prove that the circle inscribed in $\triangle ABC$ is tangent to the Euler circle constructed for that triangle at some point Φ_0 (called the *Feuerbach point*). Find the affix φ_0 of the Feuerbach point Φ_0.

7°. Prove that the three circles $(I_a), (I_b), (I_c)$ — they are escribed circles of $\triangle ABC$ at the angles A, B, C — also touch the Euler circles of $\triangle ABC$ at the points Φ_1, Φ_2, Φ_3 (also called *Feuerbach points*). Find the affixes τ_a, τ_b, τ_c of the centres I_a, I_b, I_c of the circles $(I_a), (I_b), (I_c)$. Find the affixes t_1, t_2, t_3 of the points T_1, T_2, T_3 of tangency of the circles $(I_a), (I_b), (I_c)$ to BC, CA, AB. Find the radii of these circles (Fig. 35). Find the affixes $\varphi_1, \varphi_2, \varphi_3$ of the Feuerbach points Φ_1, Φ_2, Φ_3.

Solution. 1°. The affix of the midpoint of segment EF is equal to $(z_2 + z_3)/2$ and since point A is obtained from this midpoint by inversion with respect to the unit circle (I), it follows that the affix a of point A is

$$a = \frac{2}{\bar{z}_2 + \bar{z}_3}.$$

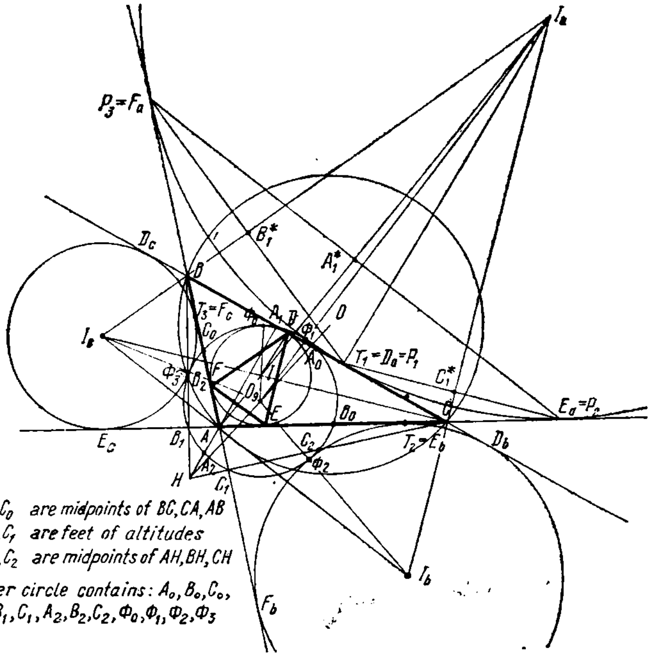

Fig. 35

Similarly, we can find the affixes b and c of points B and C:

$$b = \frac{2}{\bar{z}_3 + \bar{z}_1}, \quad c = \frac{2}{\bar{z}_1 + \bar{z}_2}.$$

The affix o of the centre O of the circle $(O) = (ABC)$ is found from the system of equations

$$(o - a)(\bar{o} - \bar{a}) = (o - b)(\bar{o} - \bar{b}),$$
$$(o - b)(\bar{o} - \bar{b}) = (o - c)(\bar{o} - \bar{c})$$

or

$$-o(\bar{a} - \bar{b}) - \bar{o}(a - b) = b\bar{b} - a\bar{a}, \tag{106}$$
$$-o(\bar{b} - \bar{c}) - \bar{o}(b - c) = c\bar{c} - b\bar{b}. \tag{107}$$

We have

$$\bar{a} - \bar{b} = \frac{2}{z_2 + z_3} - \frac{2}{z_3 + z_1} = \frac{2(z_1 - z_2)}{(z_2 + z_3)(z_3 + z_1)},$$
$$a - b = \frac{2z_2 z_3}{z_2 + z_3} - \frac{2z_3 z_1}{z_3 + z_1} = \frac{2z_3^2 (z_2 - z_1)}{(z_2 + z_3)(z_3 + z_1)},$$

$bb - a\bar{a}$

$$\frac{4z_1 z_3}{(z_3 + z_1)^2} - \frac{4z_3 z_2}{(z_2 + z_3)^2} = 4z_3 \frac{z_1(z_2^2 + 2z_2 z_3 + z_3^2) - z_2(z_3^2 + 2z_3 z_1 + z_1^2)}{(z_3 + z_1)^2 (z_2 + z_3)^2}$$

$$= 4z_3 \frac{z_1 z_2^2 + z_1 z_3^2 - z_2 z_3^2 - z_2 z_1^2}{(z_3 + z_1)^2 (z_2 + z_3)^2} = \frac{4z_3[z_1 z_2(z_2 - z_1) - z_3^2(z_2 - z_1)]}{(z_3 + z_1)^2 (z_2 + z_3)^2}$$

$$= \frac{4z_3(z_2 - z_1)(z_1 z_2 - z_3^2)}{(z_3 + z_1)^2 (z_2 + z_3)^2};$$

The equation (106) takes the form

$$o \frac{2(z_2 - z_1)}{(z_2 + z_3)(z_3 + z_1)} - \bar{o} \frac{2z_3^2(z_2 - z_1)}{(z_2 + z_3)(z_3 + z_1)} = \frac{4z_3(z_2 - z_1)(z_1 z_2 - z_3^2)}{(z_2 + z_3)^2 (z_3 + z_1)^2}$$

or

$$o - z_3^2 \bar{o} = \frac{2z_3(z_1 z_2 - z_3^2)}{(z_2 + z_3)(z_3 + z_1)}. \tag{108}$$

Similarly, equation (107) is transformed as follows:

$$o - z_1^2 \bar{o} = \frac{2z_1(z_2 z_3 - z_1^2)}{(z_3 + z_1)(z_1 + z_2)}. \tag{109}$$

Subtracting (108) from (109) term by term, we find the complex number $\bar{o}$, which is the conjugate of the affix o of point O:

$$(z_3^2 - z_1^2)\bar{o} = \frac{2z_1(z_2 z_3 - z_1^2)}{(z_1 + z_3)(z_1 + z_2)} - \frac{2z_3(z_1 z_2 - z_3^2)}{(z_2 + z_3)(z_3 + z_1)}$$

$$= \frac{(z_2 + z_3)(2\sigma_3 - 2z_1^3) - (z_1 + z_2)(2\sigma_3 - 2z_3^3)}{(z_2 + z_3)(z_3 + z_1)(z_1 + z_2)}$$

$$= \frac{2\sigma_3(z_3 - z_1) + 2z_2(z_3^3 - z_1^3) + 2z_3 z_1(z_3^2 - z_1^2)}{\sigma_1\sigma_2 - \sigma_3},$$

whence

$$(z_3 + z_1)\bar{o} = \frac{2\sigma_3 + 2z_2(z_3^2 + z_3 z_1 + z_1^2) + 2z_3 z_1(z_3 + z_1)}{\sigma_1 \sigma_2 - \sigma_3}$$

$$= \frac{4\sigma_3 + 2z_2 z_3^2 + 2z_2 z_1^2 + 2z_1 z_3^2 + 2z_3 z_1^2}{\sigma_1 \sigma_2 - \sigma_3}$$

$$= 2\frac{z_1 z_2 z_3 + z_1 z_2 z_3 + z_2 z_3^2 + z_2 z_1^2 + z_1 z_3^2 + z_3 z_1^2}{\sigma_1 \sigma_2 - \sigma_3}$$

$$= 2\frac{z_2 z_3(z_3 + z_1) + z_3 z_1(z_3 + z_1) + z_1 z_2(z_1 + z_3)}{\sigma_1 \sigma_2 - \sigma_3}.$$

Consequently,

$$\bar{o} = \frac{2\sigma_2}{\sigma_1 \sigma_2 - \sigma_3}$$

and from this we have

$$o = \frac{2\bar{\sigma}_2}{\bar{\sigma}_1 \bar{\sigma}_2 - \bar{\sigma}_3} = \frac{2\frac{\sigma_1}{\sigma_3}}{\frac{\sigma_2 \sigma_1}{\sigma_3^2} - \frac{1}{\sigma_3}} = \frac{2\sigma_1 \sigma_3}{\sigma_1 \sigma_2 - \sigma_3}.$$

2°. $R^2 = |o - a|^2 = (o - a)(\bar{o} - \bar{a}) = o\bar{o} + a\bar{a} - \bar{a}o - \bar{o}a$

$$= \frac{4\sigma_1 \sigma_2 \sigma_3}{(\sigma_1 \sigma_2 - \sigma_3)^2} + \frac{4}{(z_2 + z_3)(\bar{z}_2 + \bar{z}_3)} - \frac{4\sigma_1 \sigma_3}{\sigma_1 \sigma_2 - \sigma_3} \frac{1}{z_2 + z_3} - \frac{4\sigma_2}{\sigma_1 \sigma_2 - \sigma_3} \frac{1}{\bar{z}_2 + \bar{z}_3},$$

$$\frac{R^2}{4} = \frac{\sigma_1\sigma_2\sigma_3}{(\sigma_1 \sigma_2 - \sigma_3)^2} + \frac{z_2 z_3}{(z_2 + z_3)^2} - \frac{\sigma_1 \sigma_3}{(\sigma_1 \sigma_2 - \sigma_3)(z_2 + z_3)} - \frac{z_2 z_3 \sigma_2}{(z_2 + z_3)(\sigma_1 \sigma_2 - \sigma_3)}$$

$$= \frac{\sigma_1 \sigma_2 \sigma_3}{(\sigma_1 \sigma_2 - \sigma_3)^2} + \frac{z_2 z_3(\sigma_1 \sigma_2 - \sigma_3) - (z_2 + z_3)\sigma_1 \sigma_3 - z_2 z_3(z_2 + z_3)\sigma_2}{(z_2 + z_3)^2(\sigma_1 \sigma_2 - \sigma_3)}$$

$$= \frac{\sigma_1 \sigma_2 \sigma_3}{(\sigma_1 \sigma_2 - \sigma_3)^2} + \frac{z_2 z_3(z_2 + z_3)(z_3 + z_1)(z_1 + z_2) - (z_2 + z_3)\sigma_1\sigma_3 - z_2 z_3(z_2 + z_3)\sigma_2}{(z_2 + z_3)^2(\sigma_1 \sigma_2 - \sigma_3)}$$

$$= \frac{\sigma_1 \sigma_2 \sigma_3}{(\sigma_1 \sigma_2 - \sigma_3)^2} + \frac{z_2 z_3(z_1 + z_3)(z_1 + z_2) - \sigma_1 \sigma_3 - z_2 z_3 \sigma_2}{(z_2 + z_3)(\sigma_1 \sigma_2 - \sigma_3)}$$

$$= \frac{\sigma_1 \sigma_2 \sigma_3}{(\sigma_1 \sigma_2 - \sigma_3)^2} + \frac{z_1 \sigma_3 - \sigma_1 \sigma_3}{(z_2 + z_3)(\sigma_1 \sigma_2 - \sigma_3)} = \frac{\sigma_1 \sigma_2 \sigma_3}{(\sigma_1 \sigma_2 - \sigma_3)^2} - \frac{\sigma_3(z_2 + z_3)}{(z_2 + z_3)(\sigma_1 \sigma_2 - \sigma_3)}$$

$$= \frac{\sigma_1 \sigma_2 \sigma_3}{(\sigma_1 \sigma_2 - \sigma_3)^2} - \frac{\sigma_3}{\sigma_1 \sigma_2 - \sigma_3} = \frac{\sigma_3^2}{(\sigma_1 \sigma_2 - \sigma_3)^2}.$$

And so

$$R^2 = \frac{4\sigma_3^2}{(\sigma_1 \sigma_2 - \sigma_3)^2} = \left(\frac{2\sigma_3}{\sigma_3 - \sigma_1 \sigma_2}\right)^2.$$

We will now prove that the number

$$\lambda = \frac{\sigma_3}{\sigma_3 - \sigma_1 \sigma_2}$$

is real and positive. We have

$$\bar{\lambda} = \frac{\bar{\sigma}_3}{\bar{\sigma}_3 - \bar{\sigma}_1 \bar{\sigma}_2} = \frac{\dfrac{1}{\sigma_3}}{\dfrac{1}{\sigma_3} - \dfrac{\sigma_2 \sigma_3}{\sigma_3^2}} = \frac{\sigma_3}{\sigma_3 - \sigma_1 \sigma_2} = \lambda.$$

Hence, λ is a real number. Done differently, we have

$$\lambda = \frac{\sigma_3}{\sigma_3 - \sigma_1 \sigma_2} = \frac{1}{1 - \sigma_1 \dfrac{\sigma_2}{\sigma_3}} = \frac{1}{1 - \sigma_1 \bar{\sigma}_1} = \frac{1}{1 - |\sigma_1|^2}.$$

We now prove that $|\sigma_1| < 1$. Indeed, since all angles of $\triangle DEF$ are always acute, it follows that the orthocentre of $\triangle DEF$, the affix of which is equal to σ_1, lies inside $\triangle DEF$ and, hence, also inside the circle (DEF). But we assumed the radius of (DEF) to be 1, and so $|\sigma_1| < 1$, and, hence, $\lambda > 0$; now since $R^2 = 4\lambda^2$, it follows that $R = 2\lambda$, that is,

$$R = \frac{2\sigma_3}{\sigma_3 - \sigma_1 \sigma_2} = \frac{2}{1 - \sigma_1 \bar{\sigma}_1}.$$

3°. The radius ρ of the Euler circle of $\triangle ABC$ is

$$\rho = \frac{R}{2} = \frac{\sigma_3}{\sigma_3 - \sigma_1 \sigma_2} = \frac{1}{1 - |\sigma_1|^2} = -\frac{1}{\sigma},$$

where σ is the power of the orthocentre H' of $\triangle DEF$ with respect to the circle (DEF).

4°. Since the sum of the directed line segments

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OH},$$

where O is the centre of the circle $(O) = (ABC)$, and H is the orthocentre of $\triangle ABC$, it follows that

$$a - o + b - o + c - o = h - o,$$

whence

$$h = a + b + c - 2o,$$

where h is the affix of H. We have

$$\frac{h}{2} = \frac{z_2 z_3}{z_2 + z_3} + \frac{z_3 z_1}{z_3 + z_1} + \frac{z_1 z_2}{z_1 + z_2} - \frac{2\sigma_1 \sigma_3}{\sigma_1 \sigma_2 - \sigma_3}$$

$$= \frac{z_2 z_3(z_3 + z_1)(z_1 + z_2) + z_3 z_1(z_1 + z_2)(z_2 + z_3) + z_1 z_2(z_2 + z_3)(z_3 + z_1)}{(z_2 + z_3)(z_3 + z_1)(z_1 + z_2)} -$$

$$-\frac{2\sigma_1\sigma_3}{\sigma_1\sigma_2-\sigma_3}=\frac{z_2z_3(z_1^2+\sigma_2)+z_3z_1(z_2^2+\sigma_2)+z_1z_2(z_3^2+\sigma_2)}{(z_2+z_3)(z_3+z_1)(z_1+z_2)}-\frac{2\sigma_1\sigma_3}{\sigma_1\sigma_2-\sigma_3}$$

$$=\frac{\sigma_1\sigma_3+\sigma_2^2}{\sigma_1\sigma_2-\sigma_3}-\frac{2\sigma_1\sigma_3}{\sigma_1\sigma_2-\sigma_3}=\frac{\sigma_2^2-\sigma_1\sigma_3}{\sigma_1\sigma_2-\sigma_3}.$$

Hence,

$$h=2\frac{\sigma_2^2-\sigma_1\sigma_3}{\sigma_1\sigma_2-\sigma_3}.$$

5°. The affix ε of centre O_9 of the Euler circle of $\triangle ABC$ is equal to $\varepsilon=\frac{(h+o)}{2}$, since O_9 is the midpoint of segment OH. Thus,

$$\varepsilon=\frac{\sigma_2^2-\sigma_1\sigma_3+\sigma_1\sigma_3}{\sigma_1\sigma_2-\sigma_3}=\frac{\sigma_2^2}{\sigma_1\sigma_2-\sigma_3}.$$

6°. The equation of the radical axis of the circle (DEF) $(z\bar{z}-1=0)$ and of the Euler circle, the equation of which is

$$(z-\varepsilon)(\bar{z}-\bar{\varepsilon})-\frac{\sigma_3^2}{(\sigma_3-\sigma_1\sigma_2)^2}=0,$$

is of the form

$$z\bar{z}-1-(z-\varepsilon)(\bar{z}-\bar{\varepsilon})+\frac{\sigma_3^2}{(\sigma_3-\sigma_1\sigma_2)^2}=0$$

or

$$-1+\varepsilon\bar{z}+\bar{\varepsilon}z-\varepsilon\bar{\varepsilon}+\frac{\sigma_3^2}{(\sigma_3-\sigma_1\sigma_2)^2}=0.$$

Solving this equation together with the equation $z\bar{z}=1$ of the circle (DEF), we find

$$-1+\frac{\varepsilon}{z}+\bar{\varepsilon}z-\varepsilon\bar{\varepsilon}+\frac{\sigma_3^2}{(\sigma_3-\sigma_1\sigma_2)^2}=0, \tag{110}$$

and since

$$\bar{\varepsilon}=\frac{\dfrac{\sigma_1^2}{\sigma_3^2}}{\dfrac{\sigma_1\sigma_2}{\sigma_3^2}-\dfrac{1}{\sigma_3}}=\frac{\sigma_1^2}{\sigma_1\sigma_2-\sigma_3},$$

it follows that $\varepsilon\bar{\varepsilon}=\sigma_1^2\sigma_2^2/(\sigma_3-\sigma_1\sigma_2)^2$ and, hence,

$$-\varepsilon\bar{\varepsilon}+\frac{\sigma_3^2}{(\sigma_3-\sigma_1\sigma_2)^2}=\frac{\sigma_3^2-\sigma_1^2\sigma_2^2}{(\sigma_3-\sigma_1\sigma_2)^2}=\frac{\sigma_3+\sigma_1\sigma_2}{\sigma_3-\sigma_1\sigma_2}.$$

Equation (110) takes the form

$$\frac{\varepsilon}{z} + \bar{\varepsilon} z + \frac{\sigma_3 - \sigma_1 \sigma_2}{\sigma_3 - \sigma_1 \sigma_2} - 1 = 0$$

or

$$\frac{\varepsilon}{z} + \bar{\varepsilon} z + \frac{2\sigma_1 \sigma_2}{\sigma_3 - \sigma_1 \sigma_2} = 0$$

or

$$\bar{\varepsilon} z^2 + \frac{2\sigma_1 \sigma_2}{\sigma_3 - \sigma_1 \sigma_2} z + \varepsilon = 0. \tag{111}$$

The discriminant of this equation is equal to zero:

$$\Delta = \frac{\sigma_1^2 \sigma_2^2}{(\sigma_3 - \sigma_1 \sigma_2)^2} - \varepsilon \bar{\varepsilon} = \frac{\sigma_1^2 \sigma_2^2}{(\sigma_3 - \sigma_1 \sigma_2)^2} - \frac{\sigma_1^2 \sigma_2^2}{(\sigma_3 - \sigma_1 \sigma_2)^2} = 0.$$

Consequently, equation (111) has equal roots. This means that the radical axis of the circle (I) and of the Euler circle (O_9) of $\triangle ABC$ has a single point Φ_0 in common both with the circle (I) and the circle (O_9), that is, the circles (O_9) and (I) are tangent at the point Φ_0. The affix φ_0 of point Φ_0 (the point of tangency) is found from equation (111):

$$\varphi_0 = - \frac{\sigma_1 \sigma_2}{(\sigma_3 - \sigma_1 \sigma_2)\, \bar{\varepsilon}} = - \frac{\sigma_1 \sigma_2}{\sigma_3 - \sigma_1 \sigma_2} \cdot \frac{\sigma_1 \sigma_2 - \sigma_3}{\sigma_1^2} = \frac{\sigma_2}{\sigma_1}.$$

And so we have

$$\varphi_0 = \frac{\sigma_2}{\sigma_1}.$$

Remark. If for the unit point on the circle (DEF) we take a Boutain point of $\triangle DEF$, then $\sigma_3 = 1$ and the affix φ_0 of the Feuerbach point Φ_0 may be written thus

$$\varphi_0 = \frac{\bar{\sigma}_1}{\sigma_1}.$$

7°. We find the affixes τ_a, τ_b, τ_c of the centres of the circles $(I_a), (I_b), (I_c)$ escribed in the angles A, B, C of $\triangle ABC$. The slope of the straight line IB is equal to $b/\bar{b}$ and, hence, the slope of the straight line perpendicular to IB is equal to $-b/\bar{b}$, that is,

$$\varkappa = - \frac{b}{\bar{b}} = - \frac{z_1 + z_3}{\bar{z}_1 + \bar{z}_3} = - z_1 z_3,$$

and the equation of the bisector of the exterior angle B is of the form

$$z - \frac{2z_1 z_3}{z_1 + z_3} = - z_1 z_3\left(\bar{z} - \frac{2}{z_1 + z_3}\right)$$

or

$$z + z_1 z_2 \bar{z} = \frac{4z_1 z_3}{z_1 + z_3}. \tag{112}$$

In similar fashion we can write down the equation of the bisector of the exterior angle C:

$$z + z_1 z_2 \bar{z} = \frac{4z_1 z_2}{z_1 + z_2}. \tag{113}$$

From the system of equations (112) and (113) we find the affix $z = \tau_a$ of the point I_a:

$$(z_3 - z_2)\,\tau_a = 4\sigma_3\left(\frac{1}{z_1 + z_2} - \frac{1}{z_1 + z_3}\right),$$

$$(z_3 - z_2)\,\tau_a = 4\sigma_3\,\frac{z_3 - z_2}{(z_1 + z_2)(z_1 + z_3)},$$

whence

$$\tau_a = \frac{4\sigma_3}{(z_1 + z_2)(z_1 + z_3)}.$$

Similarly,

$$\tau_b = \frac{4\sigma_3}{(z_2 + z_1)(z_2 + z_3)}, \quad \tau_c = \frac{4\sigma_3}{(z_3 + z_1)(z_3 + z_2)}.$$

Since

$$(z_2 + z_3)(z_3 + z_1)(z_1 + z_2) = \sigma_1 \sigma_2 - \sigma_3,$$

these formulas may be rewritten thus:

$$\tau_a = \frac{4\sigma_3}{\sigma_1 \sigma_2 - \sigma_3}(z_2 + z_3) = \frac{4\sigma_3}{\sigma_1 \sigma_2 - \sigma_3}(\sigma_1 - z_1),$$

$$\tau_b = \frac{4\sigma_3}{\sigma_1 \sigma_2 - \sigma_3}(z_3 + z_1) = \frac{4\sigma_3}{\sigma_1 \sigma_2 - \sigma_3}(\sigma_1 - z_2),$$

$$\tau_c = \frac{4\sigma_3}{\sigma_1 \sigma_2 - \sigma_3}(z_1 + z_2) = \frac{4\sigma_3}{\sigma_1 \sigma_2 - \sigma_3}(\sigma_1 - z_3),$$

or thus $\left(\text{divide the numerator and the denominator of the fraction by } \sigma_3 \text{ and substitute } \bar{\sigma}_1 \text{ for } \frac{\sigma_2}{\sigma_3}\right)$:

$$\tau_a = \frac{4}{|\sigma_1|^2 - 1}(z_2 + z_3) = \frac{4}{|\sigma_1|^2 - 1}(\sigma_1 - z_1),$$
$$\tau_b = \frac{4}{|\sigma_1|^2 - 1}(z_3 + z_1) = \frac{4}{|\sigma_1|^2 - 1}(\sigma_1 - z_2),$$
$$\tau_c = \frac{4}{|\sigma_1|^2 - 1}(z_1 + z_2) = \frac{4}{|\sigma_1|^2 - 1}(\sigma_1 - z_3),$$

or, finally, thus:

$$\tau_a = -2R(\sigma_1 - z_1) = -2R\sigma_1 + 2Rz_1,$$
$$\tau_b = -2R(\sigma_1 - z_2) = -2R\sigma_1 + 2Rz_2,$$
$$\tau_c = -2R(\sigma_1 - z_3) = -2R\sigma_1 + 2Rz_3.$$

From this it follows that the centre of the circle $(I_a I_b I_c)$ has the affix

$$-2R\sigma_1 = \frac{4\sigma_1\sigma_3}{\sigma_1\sigma_2 - \sigma_3} = \frac{4\sigma_1}{\sigma_1\bar{\sigma}_1 - 1}$$

and the radius of the circle $(I_a I_b I_c)$ is equal to $2R$.

We now find the affixes t_1, t_2, t_3 of the points T_1, T_2, T_3 of contact of the circles $(I_a), (I_b), (I_c)$ with the sides BC, CA, AB respectively. Since the line segments T_1D and BC have a common midpoint, it follows that $t_1 + z_1 = b + c$, whence

$$t_1 = b + c - z_1 = \frac{2z_1z_3}{z_1 + z_3} + \frac{2z_2z_1}{z_2 + z_1} - z_1 = z_1\frac{3z_2z_3 + z_1z_3 + z_1z_2 - z_1^2}{(z_1 + z_3)(z_1 + z_2)}$$
$$= z_1\frac{2z_2z_3 + \sigma_2 - z_1^2}{(z_1 + z_2)(z_1 + z_3)} = \frac{2\sigma_3 + z_1\sigma_2 - z_1^3}{(z_1 + z_3)(z_1 + z_2)}.$$

Similarly,

$$t_2 = \frac{2\sigma_3 + z_2\sigma_2 - z_2^3}{(z_2 + z_1)(z_2 + z_3)}, \quad t_3 = \frac{2\sigma_3 + z_3\sigma_2 - z_3^3}{(z_3 + z_1)(z_3 + z_2)}.$$

We now find the radii r_a, r_b, r_c of the circles $(I_a), (I_b), (I_c)$. We first find

$$\tau_a - t_1 = \frac{4\sigma_3}{(z_1 + z_2)(z_1 + z_3)} - \frac{2\sigma_3 + z_1\sigma_2 - z_1^3}{(z_1 + z_2)(z_1 + z_3)}$$
$$= \frac{2z_1z_2z_3 - z_1(z_2z_3 + z_3z_1 + z_1z_2) + z_1^3}{(z_1 + z_2)(z_1 + z_3)}$$
$$= \frac{z_2z_3 - z_1z_3 - z_1z_2 + z_1^2}{(z_1 + z_2)(z_1 + z_3)}z_1 = \frac{z_1(z_1 - z_3) - z_2(z_1 - z_3)}{(z_1 + z_2)(z_1 + z_3)}z_1 = z_1\frac{(z_1 - z_2)(z_1 - z_3)}{(z_1 + z_2)(z_1 + z_3)}.$$

From this it follows that

$$r_a = \frac{\overrightarrow{T_1 I_a}}{\overrightarrow{ID}} = \frac{\tau_a - t_1}{z_1} = \frac{(z_1 - z_2)(z_1 - z_3)}{(z_1 + z_2)(z_1 + z_3)}$$

and, similarly,

$$r_b = \frac{(z_2 - z_1)(z_2 - z_3)}{(z_2 + z_1)(z_2 + z_3)}, \quad r_c = \frac{(z_3 - z_1)(z_3 - z_2)}{(z_3 + z_1)(z_3 + z_2)}.$$

If for the unit circle we take the circle $(I_a) = (D_a E_a F_a)$ escribed in the angle A of $\triangle ABC$ [D_a, E_a, F_a are the points of tangency of this circle (I_a) with the straight lines BC, CA, AB], then the proof of the fact that this circle touches the Euler circle constructed for $\triangle ABC$ will be precisely the same as the proof that the circles (I) and (O_9) are tangent; however, now it is necessary to assign the affixes z_1, z_2, z_3 to the points D_a, E_a, F_a. Then the affixes a, b, c of the points A, B, C will remain

$$a = \frac{2}{\bar{z}_3 + \bar{z}_2}, \quad b = \frac{2}{\bar{z}_3 + \bar{z}_1}, \quad c = \frac{2}{\bar{z}_1 + \bar{z}_2}$$

and even the affix φ_1 of the Feuerbach point Φ_1, in which the circles (I_a) and (O_9) are tangent, will be $\varphi_1 = \frac{\sigma_2}{\sigma_1}$, where σ_1 and σ_2 are expressed n terms of the affixes z_1, z_2, z_3 of the points D_a, E_a, F_a. We have to express $\varphi_1, \varphi_2, \varphi_3$ in terms of the affixes z_1, z_2, z_3 of the points D, E, F. The foregoing simplifies this problem because the fact (established above) that the circles $(I_a), (I_b), (I_c)$ are tangent to the circle (O_9) will simplify the computations (see below).

The equations of the circle (I_a) and the circle (O_9) are

$$(z - \tau_a)(\bar{z} - \bar{\tau}_a) - r_a^2 = 0$$

$$(z - \varepsilon)(\bar{z} - \bar{\varepsilon}) - \rho^2 = 0$$

By what has been proved, these circles are tangent at the Feuerbach point Φ_1, the affix of which is, consequently, found from the equation

$$\frac{r_a^2}{z - \tau_a} + \bar{\tau}_a = \frac{\rho^2}{z - \varepsilon} + \bar{\varepsilon},$$

$$\frac{r_a^2}{z - \tau_a} - \frac{\rho^2}{z - \varepsilon} + \bar{\tau}_a - \bar{\varepsilon} = 0,$$

$$r_a^2(z - \varepsilon) - \rho^2(z - \tau_a) + [z^2 - (\tau_a + \varepsilon) z + \tau_a \varepsilon](\bar{\tau}_a - \bar{\varepsilon}) = 0,$$

$$(\bar{\tau}_a - \bar{\varepsilon}) z^2 + [r_a^2 - \rho^2 - (\tau_a + \varepsilon)(\bar{\tau}_a - \bar{\varepsilon})] z - \varepsilon r_a^2 + \tau_a \rho^2 + \tau_a \varepsilon(\bar{\tau}_a - \bar{\varepsilon}) = 0.$$

Since the circles (I_a) and (O_9) have only one point in common, it follows that this quadratic equation has equal roots, which are the affixes φ_1 of the point Φ_1 of tangency of the circles (I_a) and (O_9):

$$\varphi_1 = \frac{r_a^2 - \rho^2 - (\tau_a + \varepsilon)(\bar{\tau}_a - \bar{\varepsilon})}{2(\varepsilon - \tau_a)} = \frac{r_a^2 - \rho^2}{2(\bar{\varepsilon} - \bar{\tau}_a)} + \frac{\tau_a + \varepsilon}{2}$$

$$= \frac{\dfrac{(z_1 - z_2)^2 (z_1 - z_3)^2}{(z_1 + z_2)^2 (z_1 + z_3)^2} - \dfrac{z_1^2 z_2^2 z_3^2}{(z_1 + z_2)^2 (z_2 + z_3)^2 (z_3 + z_1)^2}}{2\left[\dfrac{(z_1 + z_2 + z_3)^2}{(z_1 + z_2)(z_2 + z_3)(z_3 + z_1)} - \dfrac{4z_1(z_2 + z_3)}{(z_1 + z_2)(z_2 + z_3)(z_3 + z_1)}\right]}$$

$$+ \frac{1}{2}\left[\frac{4z_1 z_2 z_3}{(z_1 + z_2)(z_1 + z_3)} + \frac{(z_2 z_3 + z_3 z_1 + z_1 z_2)^2}{(z_1 + z_2)(z_2 + z_3)(z_3 + z_1)}\right]$$

$$= \frac{1}{2} \frac{(z_1 - z_2)^2 (z_1 - z_3)^2 (z_2 + z_3)^2 - z_1^2 z_2^2 z_3^2}{(z_1 + z_2)(z_2 + z_3)(z_3 + z_1)(z_2 + z_3 - z_1)^2}$$

$$+ \frac{1}{2} \frac{4z_1 z_2 z_3(z_2 + z_3) + (z_2 z_3 + z_3 z_1 + z_1 z_2)^2}{(z_1 + z_2)(z_2 + z_3)(z_3 + z_1)}.$$

Furthermore,

$$(z_1 - z_2)(z_1 - z_3)(z_2 + z_3) - z_1 z_2 z_3 = (z_1 - z_2 - z_3)(z_1 z_2 + z_1 z_3 - z_2 z_3),$$

$$z_2)(z_1 - z_3)(z_2 + z_3) + z_1 z_2 z_3 = (z_1 - z_2 - z_3)(z_1 z_2 + z_1 z_3 - z_2 z_3) + 2\sigma_3.$$

Hence,

$$(z_1 - z_2)^2 (z_1 - z_3)^2 (z_2 + z_3)^2 - z_1^2 z_2^2 z_3^2 = (z_2 + z_3 - z_1)^2$$
$$\times (z_1 z_2 + z_1 z_3 - z_2 z_3)^2 - 2z_1 z_2 z_3(z_2 + z_3 - z_1)(z_1 z_2 + z_1 z_3 - z_2 z_3),$$

and so

$$\varphi_1 = \frac{1}{2} \frac{(z_2 + z_3 - z_1)^2 (z_1 z_2 + z_1 z_3 - z_2 z_3)^2 - 2z_1 z_2 z_3 (z_2 + z_3 - z_1)(z_1 z_2 + z_1 z_3 - z z_3}{(z_2 + z_3)(z_3 + z_1)(z_1 + z_2)(z_2 + z_3 - z_1)^2}$$

$$+ \frac{1}{2} \frac{4z_1 z_2 z_3(z_2 + z_3) + (z_1 z_2 + z_2 z_3 + z_3 z_1)^2}{(z_2 + z_3)(z_3 + z_1)(z_1 + z_2)} = \frac{1}{2} \frac{(z_1 z_2 + z_1 z_3 - z_2 z_3)^2}{(z_2 + z_3)(z_3 + z_1)(z_1 + z_2)}$$

$$+ \frac{2z_1 z_2 z_3(z_2 + z_3)}{(z_2 + z_3)(z_3 + z_1)(z_1 + z_2)} - \frac{z_1 z_2 z_3 (z_1 z_2 + z_1 z_3 - z_2 z_3)}{(z_2 + z_3)(z_3 + z_1)(z_1 + z_2)(z_2 + z_3 - z_1)}$$

$$+ \frac{\sigma_2^2}{2(\sigma_1 \sigma_2 - \sigma_3)} = \frac{1}{2} \frac{\sigma_2^2}{\sigma_1 \sigma_2 - \sigma_3} - \frac{\sigma_3}{\sigma_1 \sigma_2 - \sigma_3} \frac{z_1 z_2 + z_1 z_3 - z_2 z_3}{z_2 + z_3 - z_1}$$

$$+ \frac{1}{2} \frac{\sigma_2^2}{\sigma_1 \sigma_2 - \sigma_3} = \frac{\sigma_2^2}{\sigma_1 \sigma_2 - \sigma_3} - \frac{\sigma_3}{\sigma_1 \sigma_2 - \sigma_3} \frac{z_1 z_2 + z_1 z_3 - z_2 z_3}{z_2 + z_3 - z_1}$$

In similar fashion we find

$$\varphi_2 = \frac{\sigma_2^2}{\sigma_1\sigma_2 - \sigma_3} - \frac{\sigma_3}{\sigma_1\sigma_2 - \sigma_3}\,\frac{z_2 z_3 + z_2 z_1 - z_3 z_1}{z_3 + z_1 - z_2},$$

$$\varphi_3 = \frac{\sigma_2^2}{\sigma_1\sigma_2 - \sigma_3} - \frac{\sigma_3}{\sigma_1\sigma_2 - \sigma_3}\,\frac{z_3 z_1 + z_3 z_2 - z_1 z_2}{z_1 + z_2 - z_3}.$$

The formulas for the affixes of the Feuerbach points are better written as follows:

$$\varphi_1 = \frac{\sigma_2^2}{\sigma_1\sigma_2 - \sigma_3} + \frac{\sigma_3}{\sigma_3 - \sigma_1\sigma_2}\,\frac{z_1 z_3 + z_1 z_2 - z_2 z_3}{z_2 + z_3 - z_1},$$

$$\varphi_2 = \frac{\sigma_2^2}{\sigma_1\sigma_2 - \sigma_3} + \frac{\sigma_3}{\sigma_3 - \sigma_1\sigma_2}\,\frac{z_2 z_1 + z_2 z_3 - z_1 z_3}{z_3 + z_1 - z_2},$$

$$\varphi_3 = \frac{\sigma_2^2}{\sigma_1\sigma_2 - \sigma_3} + \frac{\sigma_3}{\sigma_3 - \sigma_1\sigma_2}\,\frac{z_3 z_1 + z_3 z_2 - z_1 z_2}{z_1 + z_2 - z_3},$$

or as:

$$\varphi_1 = \varepsilon + \rho\,\frac{z_1 z_3 + z_1 z_2 - z_2 z_3}{z_2 + z_3 - z_1},$$

$$\varphi_2 = \varepsilon + \rho\,\frac{z_2 z_1 + z_2 z_3 - z_3 z_1}{z_3 + z_1 - z_2},$$

$$\varphi_3 = \varepsilon + \rho\,\frac{z_3 z_1 + z_3 z_2 - z_1 z_2}{z_1 + z_2 - z_3},$$

or as:

$$\varphi_1 = \varepsilon + \rho u_1, \quad \varphi_2 = \varepsilon + \rho u_2, \quad \varphi_3 = \varepsilon + \rho u_3,$$

where

$$u_1 = \frac{z_1 z_2 + z_1 z_3 - z_2 z_3}{z_2 + z_3 - z_1},$$

$$u_2 = \frac{z_2 z_1 + z_2 z_3 - z_3 z_1}{z_3 + z_1 - z_2},$$

$$u_3 = \frac{z_3 z_1 + z_3 z_2 - z_1 z_2}{z_1 + z_2 - z_3}.$$

[Note that the coefficients of ρ are complex numbers equal to unity in absolute value, for example,

$$u_1 = \frac{z_1 z_2 + z_1 z_3 - z_2 z_3}{z_2 + z_3 - z_1},$$

$$\bar{u}_1 = \frac{\dfrac{1}{z_1 z_2} + \dfrac{1}{z_1 z_3} - \dfrac{1}{z_2 z_3}}{\dfrac{1}{z_2} + \dfrac{1}{z_3} - \dfrac{1}{z_1}} = \frac{z_2 + z_3 - z_1}{z_1 z_2 + z_1 z_3 - z_2 z_3}$$

and, hence, $u_1 \bar{u}_1 = 1$. This of course immediately follows from the fact that $|\varphi_k - \varepsilon| = \rho$ $(k = 1, 2, 3)$.]

Problem 33. ABC is an arbitrary triangle; (ABC) is the circle circumscribed about it with centre O; the radius of circle (O) is equal to R; (I) is the circle inscribed in the triangle ABC; I is its centre and r the radius. Let d be the distance between the centres O and I of the circles circumscribed about and inscribed in $\triangle ABC$. Prove that

$$d^2 = R^2 - 2Rr.$$

Solution. Let D, E, F be the points of contact of the sides BC, CA, AB and the circle (I). For the unit circle, take the circle with centre I. Then the affixes of the points D, E, F, will be rz_1, rz_2, rz_3. In the preceding problem we obtained the following expression for the affix o:

$$o = \frac{2\sigma_1 \sigma_3}{\sigma_1 \sigma_2 - \sigma_3},$$

where $\sigma_1, \sigma_2, \sigma_3$ are the basic symmetric polynomials of the affixes of the vertices of the triangle. However, since these affixes are now taken in the form rz_1, rz_2, rz_3, we obtain the following expressions for the affix o of point O and for the radius R of the circle (ABC):

$$o = \frac{2\sigma_1 \sigma_3}{\sigma_1 \sigma_2 - \sigma_3} r, \quad R = \frac{2\sigma_3}{\sigma_3 - \sigma_1 \sigma_2} r.$$

Thus,

$$o = -R\sigma_1, \qquad \bar{o} = -R\bar{\sigma}_1$$

and so

$$d^2 = OI^2 = o \cdot \bar{o} = R^2 \sigma_1 \bar{\sigma}_1$$
$$= R^2 \frac{\sigma_1 \sigma_2}{\sigma_3} = R^2\left(1 - \frac{\sigma_3 - \sigma_1 \sigma_2}{\sigma_3}\right) = R^2\left(1 - \frac{2r}{R}\right) = R^2 - 2Rr.$$

Problem 34. Construct a triangle ABC if we know the points A_0, B_0, C_0 of intersection of the bisectors of the interior angles A, B, C with the circle $(O) = (ABC)$ (Fig. 36).

Solution. Take the circle (ABC) for the unit circle $(O) = (ABC) = (A_0B_0C_0)$ circumscribed about the given triangle ABC. Let z_1, z_2, z_3 be the respective affixes of the points A, B, C. The bisector of the interior angle A of $\triangle ABC$ inscribed in the circle (ABC) bisects by a point A_0 the arc $\overset{\frown}{BC}$ subtended by the chord BC; note that the points A_0 and A are located on different sides of line BC. The situation of the points B_0 and C_0 is similar (see the statement of the problem). Therefore, for the affixes

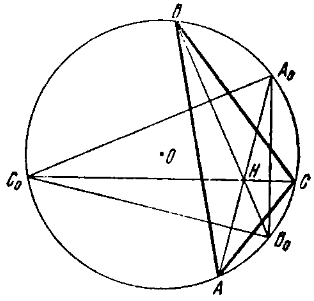

Fig. 36

$$a_0 = \sqrt{z_2 z_3}, \quad b_0 = \sqrt{z_3 z_1}, \quad c_0 = \sqrt{z_1 z_2} \quad (114)$$

of the points A_0, B_0, C_0 values must be chosen (each of the radicals has two values) so that the points A and A_0 lie on different sides of line BC, the points B and B_0 lie on different sides of line CA, and the points C and C_0 lie on different sides of line AB. It is under these conditions that we have to solve the system (114), which we rewrite as follows:

$$z_2 z_3 = a_0^2, \quad z_3 z_1 = b_0^2, \quad z_1 z_2 = c_0^2. \tag{115}$$

From the system (115) it follows that

$$z_1^2 z_2^2 z_3^2 = a_0^2 b_0^2 c_0^2$$

so that

$$z_1 z_2 z_3 = \pm a_0 b_0 c_0.$$

In the case

$$z_1 z_2 z_3 = a_0 b_0 c_0 \tag{116}$$

we have

$$z_1 = \frac{b_0 c_0}{a_0}, \quad z_2 = \frac{c_0 a_0}{b_0}, \quad z_3 = \frac{a_0 b_0}{c_0}. \tag{117}$$

In the case

$$z_1 z_2 z_3 = - a_0 b_0 c_0 \tag{118}$$

we have

$$z_1 = -\frac{b_0 c_0}{a_0}, \quad z_2 = -\frac{c_0 a_0}{b_0}, \quad z_3 = \frac{a_0 b_0}{c_0}. \tag{119}$$

Now we have to establish which of the two solutions (117) or (119) (or neither or both) constitutes a solution to the problem. Let us investigate the values of z_1, z_2, z_3 given by formulas (117). Since the foot of the perpen-

dicular dropped from point O on line BC yields the affix $\frac{z_2+z_3}{2}$, it follows that the equation of line BC may be written in the form

$$\frac{z}{\frac{z_2+z_3}{2}}+\frac{\bar{z}}{\frac{\bar{z}_2+\bar{z}_3}{2}}=2$$

or

$$\frac{z}{z_2+z_3}+\frac{\bar{z}}{\bar{z}_2+\bar{z}_3}-1=0. \tag{120}$$

We have

$$z_2+z_3=\frac{c_0 a_0}{b_0}+\frac{a_0 b_0}{c_0}=\frac{a_0}{b_0 c_0}(b_0^2+c_0^2),$$

$$\bar{z}_2+\bar{z}_3=\frac{b_0 c_0}{a_0}\left(\frac{1}{b_0^2}+\frac{1}{c_0^2}\right)=\frac{b_0^2+c_0^2}{a_0 b_0 c_0},$$

and the equation (120) of line BC takes the form

$$f(z)=\frac{b_0 c_0}{a_0(b_0^2+c_0^2)}z+\frac{a_0 b_0 c_0}{b_0^2+c_0^2}\bar{z}-1=0.$$

We find $f(a_0)$ and $f(z_1)=f\left(\frac{b_0 c_0}{a_0}\right)$:

$$f(a_0)=\frac{b_0 c_0}{b_0^2+c_0^2}+\frac{b_0 c_0}{b_0^2+c_0^2}-1=-\frac{(b_0-c_0)^2}{b_0^2+c_0^2},$$

$$f(z_1)=\frac{b_0^2 c_0^2}{a_0^2(b_0^2+c_0^2)}+\frac{a_0^2}{b_0^2+c_0^2}-1=\frac{(a_0^2-b_0^2)(a_0^2-c_0^2)}{a_0^2(b_0^2+c_0^2)}.$$

From the last two relations we have

$$\frac{f(z_1)}{f(a_0)}=-\frac{(a_0^2-b_0^2)(a_0^2-c_0^2)}{a_0^2(b_0-c_0)^2}=-\frac{(a_0-b_0)(a_0-c_0)(b_0-c_0)(a_0+b_0)(a_0+c_0)}{a_0^2(b_0-c_0)^3},$$

but

$$(A_0B_0C_0)=\frac{i}{4}\begin{vmatrix} a_0 & \bar{a}_0 & 1 \\ b_0 & \bar{b}_0 & 1 \\ c_0 & \bar{c}_0 & 1 \end{vmatrix}=\frac{i}{4a_0 b_0 c_0}\begin{vmatrix} a_0^2 & 1 & a_0 \\ b_0^2 & 1 & b_0 \\ c_0^2 & 1 & c_0 \end{vmatrix}$$

$$=\frac{i}{4a_0 b_0 c_0}(c_0-a_0)(c_0-b_0)(b_0-a_0)$$

and, consequently,

$$(c_0 - a_0)(c_0 - b_0)(b_0 - a_0) = \frac{4}{i} a_0 b_0 c_0 (A_0 B_0 C_0) = -4ia_0 b_0 c_0 (A_0 B_0 C_0).$$

Thus,

$$\frac{f(z_1)}{f(a_0)} = 4ia_0 b_0 c_0 (A_0 B_0 C_0) \frac{(a_0 + b_0)(a_0 + c_0)(b_0 - c_0)}{a_0^2 (b_0 - c_0)^4}.$$

We consider the point A_0^* with affix $a_0^* = -a_0$, which point is diametrically opposite the point A_0 on the circle $(ABC) = (A_0 B_0 C_0)$. We have

$$(a_0 + b_0)(a_0 + c_0)(b_0 - c_0) = (b_0 - a_0^*)(c_0 - a_0^*)(b_0 - c_0)$$
$$= 4ia_0 b_0 c_0 (A_0^* B_0 C_0).$$

To summarize:

$$\frac{f(z_1)}{f(a_0)} = -16(A_0 B_0 C_0)(A_0^* B_0 C_0) \frac{b_0^2 c_0^2}{(b_0 - c_0)^4}$$
$$= -16(A_0 B_0 C_0)(A_0^* B_0 C_0) \left[\frac{b_0 c_0}{(b_0 - c_0)^2}\right]^2. \quad (121)$$

The number $u = b_0 c_0/(b_0 - c_0)^2$ is a real number. Indeed,

$$\bar{u} = \frac{\dfrac{1}{b_0 c_0}}{\left(\dfrac{1}{b_0} - \dfrac{1}{c_0}\right)^2} = \frac{b_0 c_0}{(b_0 - c_0)^2} = u.$$

Therefore, $u^2 > 0$ $(u \neq 0)$. Now note that $\triangle A_0B_0C_0$, whose vertices are the points of intersection of the bisectors of the interior angles A, B, C of $\triangle ABC$ with the circle (ABC), is always an acute-angled triangle. Indeed, the interior angles A_0, B_0, C_0 of $\triangle A_0B_0C_0$ are respectively equal to

$$A_0 = \frac{B + C}{2} = \frac{\pi}{2} - \frac{A}{2},$$

$$B_0 = \frac{C + A}{2} = \frac{\pi}{2} - \frac{B}{2},$$

$$C_0 = \frac{A + B}{2} = \frac{\pi}{2} - \frac{C}{2}$$

irrespective of whether $\triangle ABC$ is acute, obtuse or right-angled.

From this it follows that the diameter $A_0A_0^*$ cuts the chord B_0C_0 and, hence, $\triangle \overrightarrow{A_0B_0C_0}$ and $\triangle \overrightarrow{A_0^*B_0C_0}$ have opposite orientations. Thus, the num-

bers $(A_0B_0C_0)$ and $(A_0^*B_0C_0)$ are of opposite sign, and from formula (121) it follows that $f(z_1)/f(a_0) > 0$, that is, the points A_0 and A lie on *one* side of the chord BC. Thus, the values (117) do not afford any solution to the problem. The required solution of system (115) is given by formulas (119), provided that $\triangle A_0B_0C_0$ is acute-angled. If it is obtuse-angled or right-angled, the problem does not have any solution.

The points A, B, C are points of intersection of the altitudes of $\triangle A_0B_0C_0$ and the circle (ABC). Indeed, the slope of line B_0C_0 is equal to $-b_0c_0$ and so the equation of the perpendicular dropped from point A_0 to the line B_0C_0 is of the form

$$z - a_0 = b_0\, c_0(\bar{z} - \bar{a}_0).$$

Solving this equation together with the equation $z\bar{z} = 1$ of the unit circle, we obtain

$$z - a_0 = b_0\, c_0 \left(\frac{1}{z} - \frac{1}{a_0}\right),$$

$$z - a_0 = -\frac{b_0\, c_0(z - a_0)}{a_0\, z}.$$

One of the roots of this equation is naturally $z = a_0$ (the affix of point A_0), the other is $z = -b_0c_0/a_0$, which is the affix of point A:

$$z_1 = -b_0\, c_0/a_0.$$

Similarly, proof can be given that

$$z_2 = -c_0a_0/b_0, \qquad z_3 = -a_0b_0/c_0,$$

which are the affixes of points B and C respectively.

Remark. From what has been proved it follows that formulas (114) should be written as follows:

$$a_0 = -\sqrt{z_2}\sqrt{z_3}, \quad b_0 = -\sqrt{z_3}\sqrt{z_1}, \quad c_0 = -\sqrt{z_1}\sqrt{z_2}, \qquad (122)$$

where we take the same value for $\sqrt{z_1}$, $\sqrt{z_2}$, $\sqrt{z_3}$ (in that case, for example, $\sqrt{z_1}\sqrt{z_1} = z_1$, $\sqrt{z_2}\sqrt{z_2} = z_2$, $\sqrt{z_3}\sqrt{z_3} = z_3$; but if for $\sqrt{z_1}$ we take different values in the last two formulas of (122), then $\sqrt{z_1}\sqrt{z_1} = -z_1$).

Problem 35. Construct $\triangle ABC$ if we are given the points A_1, B_1, C_1; these are the points of intersection of its altitudes with the circle $(O) = (ABC)$ circumscribed about $\triangle ABC$ $[(O) = (ABC) = (A_1B_1C_1)]$.

Solution. Take $(A_1B_1C_1) = (ABC)$ as the unit circle. Let a_1, b_1, c_1 be the affixes of the points A_1, B_1, C_1, and let z_1, z_2, z_3 be the affixes of the vertices A, B, C of the desired triangle ABC. Since the slope of the line AB is equal to $-z_1z_2$, it follows that the equation of the altitude dropped from vertex C on the side AB is of the form

$$z - z_3 = z_1\, z_2(\bar{z} - \bar{z}_3).$$

Solving this equation together with the equation $z\bar{z} = 1$ of the unit circle, we obtain

$$z - z_3 = z_1 z_2 \left(\frac{1}{z} - \frac{1}{z_3} \right),$$

$$z - z_3 = - \frac{z_1 z_2}{z_3 z} (z - z_3).$$

One of the roots of this equation is naturally equal to $z = z_3$ (the affix of point C). The other is

$$z = c_1 = - z_1 z_2 / z_3, \tag{123}$$

which is the affix of point C_1. In similar fashion we find the affixes

$$a_1 = - z_2 z_3 / z_1, \qquad b_1 = - z_3 z_1 / z_2 \tag{124}$$

of the points B_1 and A_1.

Let us solve the resulting system for z_1, z_2, z_3. Multiplying these equations together pairwise, we obtain

$$z_1^2 = b_1 c_1, \quad z_2^2 = c_1 a_1, \quad z_3^2 = a_1 b_1,$$

whence

$$z_1 = \sqrt{b_1} \sqrt{c_1}, \quad z_2 = \sqrt{c_1} \sqrt{a_1}, \quad z_3 = \sqrt{a_1} \sqrt{b_1}, \tag{125}$$

and since each of the products of two radicals in the right-hand member of each equation has two values, the system (125) has 8 solutions. Assuming that in each of the equations (125), any value is taken for each radical $\sqrt{a_1}$, $\sqrt{b_1}$, $\sqrt{c_1}$ (but one and the same value in each equation), we can write all the solutions as follows:

$$\left.\begin{aligned}
z_1 &= \sqrt{b_1}\sqrt{c_1}, & z_2 &= \sqrt{c_1}\sqrt{a_1}, & z_3 &= \sqrt{a_1}\sqrt{b_1},\\
z_1 &= -\sqrt{b_1}\sqrt{c_1}, & z_2 &= -\sqrt{c_1}\sqrt{a_1}, & z_3 &= \sqrt{a_1}\sqrt{b_1},\\
z_1 &= \sqrt{b_1}\sqrt{c_1}, & z_2 &= -\sqrt{c_1}\sqrt{a_1}, & z_3 &= -\sqrt{a_1}\sqrt{b_1},\\
z_1 &= -\sqrt{b_1}\sqrt{c_1}, & z_2 &= \sqrt{c_1}\sqrt{a_1}, & z_3 &= -\sqrt{a_1}\sqrt{b_1};
\end{aligned}\right\} \tag{126}$$

$$\left.\begin{aligned}
z_1 &= -\sqrt{b_1}\sqrt{c_1}, & z_2 &= -\sqrt{c_1}\sqrt{a_1}, & z_3 &= -\sqrt{a_1}\sqrt{b_1},\\
z_1 &= -\sqrt{b_1}\sqrt{c_1}, & z_2 &= \sqrt{c_1}\sqrt{a_1}, & z_3 &= \sqrt{a_1}\sqrt{b_1},\\
z_1 &= \sqrt{b_1}\sqrt{c_1}, & z_2 &= -\sqrt{c_1}\sqrt{a_1}, & z_3 &= \sqrt{a_1}\sqrt{b_1},\\
z_1 &= \sqrt{b_1}\sqrt{c_1}, & z_2 &= \sqrt{c_1}\sqrt{a_1}, & z_3 &= -\sqrt{a_1}\sqrt{b_1}.
\end{aligned}\right\} \tag{127}$$

From the system (123), (124) it follows that $a_1b_1c_1 = -z_1z_2z_3$ and so all number triples (126) fail to serve as solutions to the system (124). Now any row of relations (127) is a solution of the system (123), (124). Indeed,

$$-\frac{z_1 z_2}{z_3} = -\frac{\sqrt{b_1}\sqrt{c_1}\sqrt{c_1}\sqrt{a_1}}{-\sqrt{a_1}\sqrt{b_1}} = c_1,$$

$$-\frac{z_2 z_3}{z_1} = -\frac{\sqrt{c_1}\sqrt{a_1}\sqrt{a_1}\sqrt{b_1}}{-\sqrt{b_1}\sqrt{c_1}} = a_1,$$

$$-\frac{z_3 z_1}{z_2} = -\frac{\sqrt{a_1}\sqrt{b_1}\sqrt{b_1}\sqrt{c_1}}{-\sqrt{c_1}\sqrt{a_1}} = b_1$$

and also for the other three relations (127).

To summarize, then, there exist four triangles that satisfy the condition of the problem. Let us construct, for example, a triangle corresponding to the first row of solutions (127). To do this, draw to the circle $(A_1B_1C_1)$ any one of two tangent lines parallel to the straight line ΩA_1 (Ω is the unit point); the point P_1, which is the point of contact of the tangent drawn to the circle $(A_1B_1C_1)$, has as its affix one of the values $\sqrt{a_1}$ (Fig. 37). In similar fashion, construct the points Q_1 and R_1, whose affixes are the values $\sqrt{b_1}$ and $\sqrt{c_1}$. To construct a point with the affix $\sqrt{b_1}\sqrt{c_1}$, draw through the point Ω a straight line parallel to the line Q_1R_1; the second point P, the point of intersection of the drawn line and the circle $(A_1B_1C_1)$, has the affix $\sqrt{b_1}\sqrt{c_1}$. Finally, point A, which is symmetric to point P with respect to the centre O of $(A_1B_1C_1)$ has the affix $-\sqrt{b_1}\sqrt{c_1}$. The points B and C with affixes $-\sqrt{c_1}\sqrt{a_1}$ and $-\sqrt{a_1}\sqrt{b_1}$ are constructed in similar fashion.

The other three triangles that satisfy the conditions of the problem are: AQR, PBR, PQC.

Problem 36. Inscribed in the unit circle is $\triangle ABC$, the affixes of whose vertices are z_1, z_2, z_3. Find the affixes $\tau_0, \tau_1, \tau_2, \tau_3$ of the centre of the circle (I) inscribed in that triangle (affix τ_0) and of the centres of the circles (I_a), (I_b), (I_c) escribed in that triangle (in the angles A, B, C, respectively).

Solution. The centre I of the circle (I) inscribed in $\triangle ABC$ is the point of intersection of the bisectors of the interior angles. These bisectors intersect the circle (ABC) in the points A_0, B_0, C_0; note that $\triangle A_0B_0C_0$ is always an obtuse-angled triangle (see problem 34). The affixes a_0, b_0, c_0, of the points A_0, B_0, C_0 are expressed by the formulas (122) of problem 34:

$$a_0 = -\sqrt{z_2}\sqrt{z_3}, \quad b_0 = -\sqrt{z_3}\sqrt{z_1}, \quad c_0 = -\sqrt{z_1}\sqrt{z_2}, \qquad (128)$$

where the same values are taken in all formulas for the square roots $\sqrt{z_1}$, $\sqrt{z_2}$, $\sqrt{z_3}$; note that these values for $\sqrt{z_1}$, $\sqrt{z_2}$, $\sqrt{z_3}$ must always be taken so that formulas (128) define just the points of intersection of the bisectors

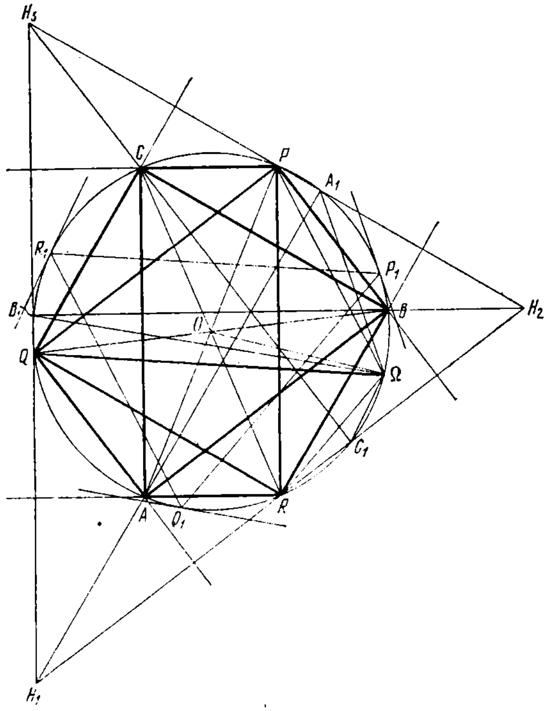

Fig. 37

of the interior angles A, B, C of the given $\triangle ABC$ with (ABC). Namely, for $\sqrt{z_1}$, $\sqrt{z_2}$, $\sqrt{z_3}$ we have to take the values so that the inequality $|\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}| < 1$ holds, that is, so that the triangle whose vertices have the affixes $\sqrt{z_1}$, $\sqrt{z_2}$, $\sqrt{z_3}$ is acute-angled. Indeed, then

$$
\begin{aligned}
|a_0 + b_0 + c_0)| &= |-\sqrt{z_2}\sqrt{z_3} - \sqrt{z_3}\sqrt{z_1} - \sqrt{z_1}\sqrt{z_2}| \\
&= |\sqrt{z_2}\sqrt{z_3} + \sqrt{z_3}\sqrt{z_1} + \sqrt{z_1}\sqrt{z_2}| \\
&= |\overline{\sqrt{z_2}}\,\overline{\sqrt{z_3}} + \overline{\sqrt{z_3}}\,\overline{\sqrt{z_1}} + \overline{\sqrt{z_1}}\,\overline{\sqrt{z_2}}|
\end{aligned}
$$

$$
= \left|\frac{1}{\sqrt{z_2}\sqrt{z_3}} + \frac{1}{\sqrt{z_3}\sqrt{z_1}} + \frac{1}{\sqrt{z_1}\sqrt{z_2}}\right| = \frac{|\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}|}{|\sqrt{z_1}\sqrt{z_2}\sqrt{z_3}|}
$$

$$
= |\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}| < 1.
$$

That is, $\triangle A_0B_0C_0$ is acute-angled [there are only two such choices for the square roots of z_1, z_2, z_3; if one of them is denoted by $\sqrt{z_1}$, $\sqrt{z_2}$, $\sqrt{z_3}$, then the other will be $-\sqrt{z_1}$, $-\sqrt{z_2}$, $-\sqrt{z_3}$, and any one of these choices yields the formulas (128)].

To summarize, then, AA_0, BB_0, CC_0 are the bisectors of the interior angles A, B, C of $\triangle ABC$. The slope of the bisector AA_0 is equal to

$$\frac{a_0 - z_1}{\bar{a}_0 - \bar{z}_1} = \frac{-\sqrt{z_2}\sqrt{z_3} - z_1}{-\dfrac{1}{\sqrt{z_2}\sqrt{z_3}} - \dfrac{1}{z_1}} = z_1\sqrt{z_2}\sqrt{z_3},$$

and, hence, the equation of the straight line AA_0 is

$$z - z_1 = z_1\sqrt{z_2}\sqrt{z_3}\,(\bar{z} - \bar{z}_1)$$

or

$$z - z_1\sqrt{z_2}\sqrt{z_3}\,\bar{z} = z_1 - \sqrt{z_2}\sqrt{z_3}. \tag{129}$$

The equation of BB_0 is

$$z - z_2\sqrt{z_3}\sqrt{z_1}\,\bar{z} = z_2 - \sqrt{z_3}\sqrt{z_1}. \tag{130}$$

From the system (129), (130) we find the affix τ_0 of the centre I of the circle (I) inscribed in $\triangle ABC$:

$$\tau_0(z_2\sqrt{z_1} - z_1\sqrt{z_2}) = -z_1z_2(\sqrt{z_2} - \sqrt{z_1}) - \sqrt{z_1}\sqrt{z_2}\sqrt{z_3}\,(z_2 - z_1),$$

or

$$\tau_0\sqrt{z_1}\sqrt{z_2}(\sqrt{z_2} - \sqrt{z_1}) = -z_1z_2(\sqrt{z_2} - \sqrt{z_1}) - \sqrt{z_1}\sqrt{z_2}\sqrt{z_3}\,[(\sqrt{z_2})^2 - (\sqrt{z_1})^2]$$

That is,

$$\tau_0 = -\sqrt{z_1}\sqrt{z_2} - \sqrt{z_3}(\sqrt{z_2} + \sqrt{z_1}),$$

and, finally,

$$\tau_0 = -\sqrt{z_2}\sqrt{z_3} - \sqrt{z_3}\sqrt{z_1} - \sqrt{z_1}\sqrt{z_2} = a_0 + b_0 + c_0. \tag{131}$$

If we now take the point B_0^*, which is symmetric to the point B_0 with respect to the centre O of the circle (ABC), then the affix of the point B_0^* will be equal to $b_0^* = -b_0$ and BB_0^* will be the bisector of the exterior angle of the triangle at the vertex B. The slope of BB_0^* is

$$\frac{z_2 + b_0}{\bar{z}_2 + \bar{b}_0} = \frac{z_2 - \sqrt{z_3}\sqrt{z_1}}{\dfrac{1}{z_2} - \dfrac{1}{\sqrt{z_3}\sqrt{z_1}}} = -z_2\sqrt{z_3}\sqrt{z_1},$$

and the equation of line BB_0^* is

$$z - z_2 = -z_2\sqrt{z_3}\sqrt{z_1}\,(\bar{z} - \bar{z}_2)$$

or

$$z + z_2\sqrt{z_3}\sqrt{z_1}\,\bar{z} = z_2 + \sqrt{z_3}\sqrt{z_1}. \tag{132}$$

From the equations (129) and (132) we find the affix τ_1 of the centre I_a of the circle (I_a) escribed in angle A of triangle ABC:

$$\tau_1 (z_1 \sqrt{z_2} + z_2 \sqrt{z_1}) = z_2 z_1 (\sqrt{z_2} + \sqrt{z_1}) - \sqrt{z_1}\sqrt{z_2}\sqrt{z_3}(z_2 - z_1)$$

or

$$\tau_1 \sqrt{z_1}\sqrt{z_2}(\sqrt{z_1} + \sqrt{z_2}) = z_1 z_2 (\sqrt{z_1} + \sqrt{z_2}) - \sqrt{z_1}\sqrt{z_2}\sqrt{z_3}[(\sqrt{z_2})^2 - (\sqrt{z_1})^2],$$

that is,

$$\tau_1 = \sqrt{z_1}\sqrt{z_2} - \sqrt{z_3}(\sqrt{z_2} - \sqrt{z_1}) = \sqrt{z_1}\sqrt{z_2} - \sqrt{z_3}\sqrt{z_2} + \sqrt{z_3}\sqrt{z_1},$$

and, finally,

$$\tau_1 = -\sqrt{z_2}\sqrt{z_3} + \sqrt{z_3}\sqrt{z_1} + \sqrt{z_1}\sqrt{z_2} = a_0 - b_0 - c_0.$$

Similarly we find τ_2 and τ_3. Thus,

$$\left.\begin{aligned}
\tau_0 &= -\sqrt{z_2}\sqrt{z_3} - \sqrt{z_3}\sqrt{z_1} - \sqrt{z_1}\sqrt{z_2} = a_0 + b_0 + c_0,\\
\tau_1 &= -\sqrt{z_2}\sqrt{z_3} + \sqrt{z_3}\sqrt{z_1} + \sqrt{z_1}\sqrt{z_2} = a_0 - b_0 - c_0,\\
\tau_2 &= \sqrt{z_2}\sqrt{z_3} - \sqrt{z_3}\sqrt{z_1} + \sqrt{z_1}\sqrt{z_2} = -a_0 + b_0 - c_0,\\
\tau_3 &= \sqrt{z_3}\sqrt{z_3} + \sqrt{z_3}\sqrt{z_1} - \sqrt{z_1}\sqrt{z_2} = -a_0 - b_0 + c_0.
\end{aligned}\right\} \qquad (133)$$

From formulas (133) it follows that

$$\frac{\tau_0 + \tau_1}{2} = a_0, \quad \frac{\tau_0 + \tau_2}{2} = b_0, \quad \frac{\tau_0 + \tau_3}{2} = c_0.$$

That is, the midpoints of the segments II_a, II_b, II_c coincide respectively with the points A_0, B_0, C_0 in which the bisectors of the interior angles of $\triangle ABC$ intersect the circle $(ABC) = (O)$ circumscribed about $\triangle ABC$ (Fig. 38).

From formulas (133) it also follows that

$$\frac{\tau_2 + \tau_3}{2} = -a_0, \quad \frac{\tau_3 + \tau_1}{2} = -b_0, \quad \frac{\tau_1 + \tau_2}{2} = -c_0,$$

that is, the midpoints of the segments I_bI_c, I_cI_a, I_aI_b coincide respectively with the points A_0^*, B_0^*, C_0^* in which the bisectors of the exterior angles A, B, C of $\triangle ABC$ cuts the circle $(ABC) = (O)$ circumscribed about $\triangle ABC$. The points A_0^*, B_0^*, C_0^* are symmetric respectively to the points A_0, B_0, C_0 about the centre O of $(O) = (ABC)$ so that $A_0A_0^*$, $B_0B_0^*$, $C_0C_0^*$ are diameters of that circle. Incidentally, from that it follows that the circle $(O) = (ABC) = (A_0B_0C_0) = (A_0^*B_0^*C_0^*)$ is the Euler circle of $\triangle I_aI_bI_c$ and therefore the radius of the circle $(I_aI_bI_c)$ is twice the radius of the circle (ABC) [this has already been proved analytically in problem 32, item 7° (see solution)].

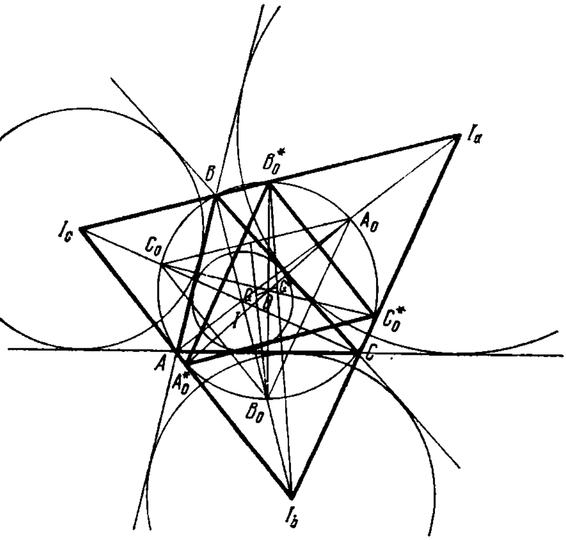

Fig. 38

Finally, since $IA \perp I_bI_c$, $IB \perp I_cI_a$, $IC \perp I_aI_b$, it follows that I is the orthocentre of $\triangle I_aI_bI_c$; furthermore, since $(\tau_1 + \tau_2 + \tau_3)/3 = -(a_0 + b_0 + c_0)/3$, it follows that the centroid G^* of $\triangle I_aI_bI_c$ is symmetric to the centroid G of $\triangle A_0B_0C_0$ about the centre O of the circle (ABC), or the point G^* is the centroid of $\triangle A_0^*B_0^*C_0^*$, which is symmetric to $\triangle A_0B_0C_0$ with respect to the point O.

Problem 37. ABC is an arbitrary triangle and P is an arbitrary point. Prove that the straight lines a', b', c', which are symmetric to the straight lines $a = AP$, $b = BP$, $c = CP$ about the bisectors AI, BI, CI of the interior angles A, B, C of $\triangle ABC$ also pass through one and the same point Q. The points P and Q are said to be *isogonally conjugate* with respect to the triangle ABC (Fig. 39). Prove that if the circle (ABC) is taken as the unit circle and if the affixes of the points A, B, C, P, Q are equal respectively to z_1, z_2, z_3, p, q, then they are connected by the relation

$$p + q + \sigma_3\bar{p}\,\bar{q} = \sigma_1 \tag{134}$$

(this relation was obtained by the English mathematician Morley). Express q in terms of z_1, z_2, z_3, p.

Solution. A_0, B_0, C_0, the points of intersection of the bisectors of the interior angles A, B, C of $\triangle ABC$ with the circle (ABC), always form an acute-angled triangle $A_0B_0C_0$. If for $\sqrt{z_1}, \sqrt{z_2}, \sqrt{z_3}$ we choose values such that $|\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}| < 1$, then the affixes a_0, b_0, c_0 of points A_0, B_0, C_0 are

$$a_0 = -\sqrt{z_2}\sqrt{z_3}, \quad b_0 = -\sqrt{z_3}\sqrt{z_1}, \quad c_0 = -\sqrt{z_1}\sqrt{z_2}$$

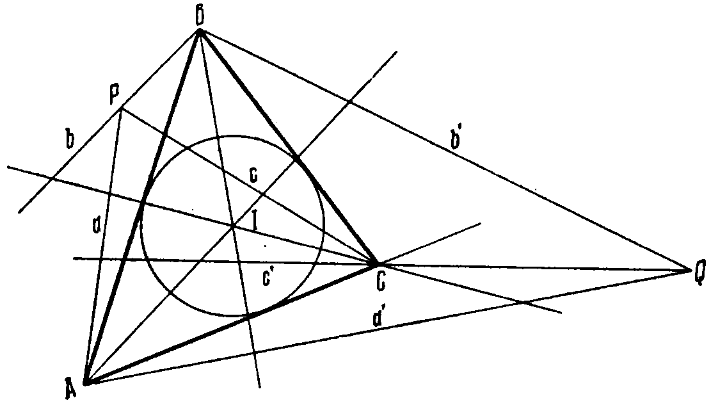

Fig. 39

[problem 36, formulas (128)]. The slope of the bisector of the interior angle A of $\triangle ABC$ is

$$\frac{z_1 + \sqrt{z_2}\sqrt{z_3}}{\bar{z}_1 + \overline{\sqrt{z_2}}\,\overline{\sqrt{z_3}}} = z_1\sqrt{z_2}\sqrt{z_3}$$

and the equation of this bisector is

$$z - z_1\sqrt{z_2}\sqrt{z_3}\,\bar{z} = z_1 - \sqrt{z_2}\sqrt{z_3} \tag{135}$$

[see problem 36, equation (129)].

The equation of the perpendicular dropped from point P on this line is of the form

$$z - p = -z_1\sqrt{z_2}\sqrt{z_3}\,(\bar{z} - \bar{p})$$

or

$$z + z_1\sqrt{z_2}\sqrt{z_3}\,\bar{z} = p + z_1\sqrt{z_2}\sqrt{z_3}\,\bar{p}. \tag{136}$$

Adding the equations (135) and (136) term by term, we find the affix $z = p'$ of projection P' of point P on the bisector of the interior angle A of $\triangle ABC$:

$$p' = \frac{1}{2}\left(z_1 + p - \sqrt{z_2}\sqrt{z_3} + z_1\sqrt{z_2}\sqrt{z_3}\,\bar{p}\right).$$

The affix p_1^* of the point P_1^*, which is symmetric to point P about the bisector AI of the interior angle A of $\triangle ABC$ is found from the relation

$$\frac{p + p_1^*}{2} = p' = \frac{1}{2}\left(z_1 + p - \sqrt{z_2}\sqrt{z_3} + z_1\sqrt{z_2}\sqrt{z_3}\,\bar{p}\right),$$

whence

$$p_1^* = z_1 - \sqrt{z_2}\sqrt{z_3} + z_1\sqrt{z_2}\sqrt{z_3}\,\bar{p}.$$

The equation of the line AP_1^* may be written thus:

$$\begin{vmatrix} z & z & p_1^* \\ \bar{z} & \bar{z}_1 & \bar{p}_1^* \\ 1 & 1 & 1 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} z & z_1 & z_1 - \sqrt{z_2}\sqrt{z_3} + z_1\sqrt{z_2}\sqrt{z_2}\,\bar{p} \\ \bar{z} & \bar{z}_1 & \bar{z}_1 - \overline{\sqrt{z_2}}\,\overline{\sqrt{z_3}} + \bar{z}_1\overline{\sqrt{z_2}}\,\overline{\sqrt{z_3}}\,p \\ 1 & 1 & 1 \end{vmatrix} = 0$$

or

$$\begin{aligned}(\overline{\sqrt{z_2}}\,\overline{\sqrt{z_3}} - \bar{z}_1\overline{\sqrt{z_2}}\,\overline{\sqrt{z_3}}\,p)\,z + (-\sqrt{z_2}\sqrt{z_3} + z_1\sqrt{z_2}\sqrt{z_3}\,\bar{p})\,\bar{z}& \\ - z_1\overline{\sqrt{z_2}}\,\overline{\sqrt{z_3}} + \overline{\sqrt{z_2}}\,\overline{\sqrt{z_3}}\,p + \bar{z}_1\sqrt{z_2}\sqrt{z_3} - \sqrt{z_2}\sqrt{z_3}\,\bar{p} &= 0\end{aligned}$$

or, multiplying both sides of the last equation by $\sqrt{z_2}\sqrt{z_3}$,

$$(1 - \bar{z}_1 p)z + (-z_2z_3 + z_1z_2z_3\bar{p})\bar{z} - z_1 + p + \bar{z}_1z_2z_3 - z_2z_3\bar{p} = 0. \quad (137)$$

In similar fashion we can write down the equation of the straight line symmetric to line BP about the bisector BI of the interior angle B of $\triangle ABC$:

$$(1 - \bar{z}_2 p)\,z + (-z_3z_1 + z_1z_2z_3\bar{p})\,\bar{z} - z_2 + p + \bar{z}_2z_3z_1 - z_3z_1\bar{p} = 0. \quad (138)$$

The affix of point Q, the point of intersection of the lines (137) and (138) that are symmetric to the lines AP and BP about the bisectors AI and BI of the interior angles A and B of $\triangle ABC$, may be found by solving the system of equations (137) and (138). Since

$$\begin{aligned}\Delta = \begin{vmatrix} 1 - \bar{z}_1 p & -z_2z_3 + z_1z_2z_3\bar{p} \\ 1 - \bar{z}_2 p & -z_3z_1 + z_1z_2z_3\bar{p} \end{vmatrix} &= -z_3z_1 + \sigma_3\bar{p} + z_3p - z_2z_3p\bar{p} + z_2z_3 - \sigma_3\bar{p} - z_3p + z_1z_3\bar{p}p \\ &= z_3(z_2 - z_1) - z_3p\bar{p}\,(z_2 - z_1) = z_3\,(z_2 - z_1)\,(1 - p\bar{p}).\end{aligned}$$

it follows that $\Delta \neq 0$ if and only if $1 - p\bar{p} \neq 0$, that is $|p| \neq 1$; in other words, when the point P does not lie on the circle (ABC).

(1) Assuming that $|p| \neq 1$, that is that the point P does not lie on the circle (ABC), we conclude that the system (137), (138) has a unique solution for z and $\bar{z}$: $z = q$, $\bar{z} = \bar{q}$. This solution could be found, for example, via the Cramer formulas, but we will take a somewhat different route; namely, by substituting the value $z = q$, $\bar{z} = \bar{q}$ into the equations (137),

(138) and subtracting the second equation from the first term by term, we get

$$(\bar{z}_2 - \bar{z}_1)pq + z_3(z_1 - z_2)\bar{q} + z_2 - z_1 + z_3\left(\frac{z_2}{z_1} - \frac{z_1}{z_2}\right) + z_3\bar{p}(z_1 - z_2) = 0$$

or

$$\frac{z_1 - z_2}{z_1 z_2} pq + z_3\bar{q}(z_1 - z_2) - (z_1 - z_2) - \frac{z_3}{z_1 z_2}(z_1^2 - z_2^2) + z_3\bar{p}(z_1 - z_2) = 0.$$

Cancelling $z_1 - z_2$ and multiplying both sides by $z_1 z_2$, we obtain

$$pq + \sigma_3(\bar{p} + \bar{q}) - \sigma_2 = 0. \tag{139}$$

Now this is the *Morley relation*. Indeed, passing to conjugate numbers, we have

$$\bar{p}\bar{q} + \bar{\sigma}_3(p + q) - \bar{\sigma}_2 = 0. \tag{140}$$

But $\bar{\sigma}_3 = 1/\sigma_3$, $\bar{\sigma}_2 = \sigma_1/\sigma_3$ and so

$$p + q + \sigma_3\bar{p}\bar{q} = \sigma_1. \tag{141}$$

From the relations (139) and (141) it is easy to express q in terms of p. From (139) we find

$$\bar{q} = \frac{\sigma_2 - pq}{\sigma_3} - \bar{p}$$

and (141) takes the form

$$p + q + \bar{p}(\sigma_2 - pq - \sigma_3\bar{p}) = \sigma_1$$

or

$$q(1 - p\bar{p}) = \sigma_3\bar{p}^2 - \sigma_2\bar{p} - p + \sigma_1,$$

whence

$$q = \frac{\sigma_3\bar{p}^2 - \sigma_2\bar{p} - p + \sigma_1}{1 - p\bar{p}}.$$

From the symmetry of this expression with respect to z_1, z_2, z_3 it follows that the straight line C' as well passes through the point Q.

(2) If point P lies on the circle (ABC) but does not coincide with any of the points A, B, C, then the straight lines (137) and (138) are collinear; as will be evident from what follows, there is a third line collinear with them: it is the line symmetric to line CP with respect to the straight line CI (Fig. 40). The slopes of these lines are:

$$\varkappa = \frac{z_2 z_3 - \sigma_3\bar{p}}{1 - \bar{z}_1 p} = \sigma_3\frac{\bar{z}_1 - \bar{p}}{1 - \bar{z}_1 p} = \sigma_3\frac{\frac{1}{z_1} - \frac{1}{p}}{1 - \frac{p}{z}} = \sigma_3\frac{p - z_1}{p(z_1 - p)} = -\frac{\sigma_3}{p} = -\sigma_3\bar{p},$$

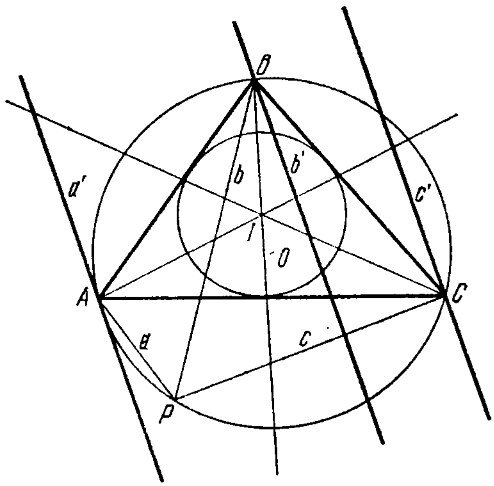

Fig. 40

and if we take a Boutain point for the unit point, then $\varkappa = -\bar{p}$. Setting $p = \cos\ \alpha + i \sin\ \alpha$, we find

$$\varkappa = \cos(\pi - \alpha) + i \sin(\pi - \alpha).$$

The affixes of the ends of a diameter of the unit circle (ABC), which has such a slope, are

$$\sqrt{\varkappa} = \pm \left[\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + i \sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right].$$

From this it follows that if the point P describes the circle (ABC), then this diameter will turn through an angle π in a direction opposite that of the radius OP (Fig. 41). If the Euclidean plane is completed to a projective-Euclidean plane by points at infinity, or ideal points, we can then assume that if the point P lies on (ABC) but does not coincide with any one of the vertices A, B, C of $\triangle ABC$, then it will be associated with the ideal point of the projective-Euclidean plane. The Morley relation $p + q + \sigma_3 \bar{p}\bar{q} = \sigma_1$ does not, of course, hold in this case since no complex number can be associated with ideal points.

Finally, from geometric reasoning it is clear that if point P lies on line BC but does not coincide with B or C, then point A will be isogonally conjugate to it (the same goes for the lines CA and AB). Finally, if point P coincides with one of the vertices of $\triangle ABC$, for example

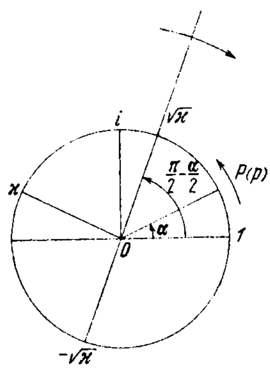

Fig. 41

$P = A$, then *any* point of BC will be isogonally conjugate to it (and the same goes for the cases $P = B$ and $P = C$). This can also be verified analytically: let $p = \alpha z_2 + \beta z_3$, where α and β are real and $\alpha + \beta = 1$ (that is, the point P lies on the straight line BC). Then $\bar{p} = \frac{\alpha}{z_2} + \frac{\beta}{z_3}$ and, assuming $q = z_1$, $\bar{q} = \frac{1}{z_1}$, we will have

$$p + q + \sigma_3 \bar{p}\,\bar{q} = \alpha z_2 + \beta z_3 + z_1 + z_1 z_2 z_3 \left(\frac{\alpha}{z_2} + \frac{\beta}{z_3}\right)\frac{1}{z_1}$$

$$= \alpha z_2 + \beta z_3 + z_1 + \alpha z_3 + \beta z_2 = z_1 + (\alpha + \beta) z_2 + (\alpha + \beta) z_3$$

$$= z_1 + z_2 + z_3 = \sigma_1.$$

Note that the centre I of the circle (I) inscribed in $\triangle\, ABC$, and the centres I_a, I_b, I_c of the circles (I_a), (I_b), (I_c) escribed in the angles A, B, C of that triangle are isogonal conjugates of themselves with respect to $\triangle\, ABC$. It will be proved below, in problem 41, that besides the points I, I_a, I_b, I_c there are no points that are isogonal conjugates of themselves with respect to $\triangle\, ABC$.

We also note that if we eliminate from the Euclidean plane the circle (ABC) circumscribed about $\triangle\, ABC$ and eliminate the lines BC, CA, AB, then the isogonal correspondence between the points $P(p)$ and $Q(q)$ will be a one-to-one mapping $(P \longleftrightarrow Q)$ or a one-to-one and involutory transformation (involution) described by the relation $p + q + \bar{p}\,\bar{q}\sigma_3 = \sigma_1$.

If the Euclidean plane is completed to a projective-Euclidean plane, and the straight lines BC, CA, AB are deleted, then the correspondence between the points $P(p)$ and $Q(q)$ will again be one-to-one, but then the relation $p + q + \bar{p}\bar{q}\sigma_3 = \sigma_1$ will refer only to the ideal points $P(p)$ and $Q(q)$. Finally, throughout the projective-Euclidean plane, the mapping of $P(p)$ on $Q(q)$ will no longer be a one-to-one mapping and the relation $p + q + \bar{p}\bar{q}\sigma_3 = \sigma_1$ will again be valid only for the proper points $P(p)$ and $Q(q)$ (ideal points do not have affixes).

Problem 38. The circle $(O) = (ABC)$ circumscribed about $\triangle\, ABC$ is taken as the unit circle; z_1, z_2, z_3 are the affixes of the vertices A, B, C. Find the affixes of the points that are isogonal conjugates of the following points with respect to $\triangle\, ABC$:

1°. The point G, the point of intersection of the medians of $\triangle\, ABC$.

2°. The orthocentre H of $\triangle\, ABC$.

3°. The centre O_9 of the Euler circle of $\triangle\, ABC$.

Solution. Let us take advantage of the formula of problem 37;

$$q = \frac{p + \bar{p}\sigma_2 - \sigma_1 - \sigma_3 \bar{p}^2}{p\bar{p} - 1},$$

which defines the affix q of point Q; this is the image of point P with affix p under an isogonal transformation with respect to $\triangle ABC$. We have the following.

1°. Since the affix g of point G is equal to $\sigma_1/3$, it follows that the affix l of the image L of point G under an isogonal transformation with respect to $\triangle ABC$ is

$$l = \frac{\dfrac{\sigma_1}{3} + \dfrac{\bar\sigma_1}{3}\sigma_2 - \sigma_1 - \sigma_3\dfrac{\bar\sigma_1^2}{9}}{\dfrac{\sigma_1\bar\sigma_1}{9} - 1} = \frac{\dfrac{2}{3}\sigma_1 + \dfrac{\sigma_2^2}{3\sigma_3} - \dfrac{\sigma_2^2}{9\sigma_3}}{\dfrac{\sigma_1\bar\sigma_1}{9} - 1}$$

$$= \frac{-\dfrac{2}{3}\sigma_1 + \dfrac{2\sigma_2^2}{9\sigma_3}}{\dfrac{\sigma_1\bar\sigma_1}{9} - 1} = 2\,\frac{\sigma_2\bar\sigma_1 - 3\sigma_1}{\sigma_1\bar\sigma_1 - 9}.$$

The point L, which is the isogonal conjugate of point G with respect to $\triangle ABC$, is termed the *Lemoine point*. Thus, the affix l of the Lemoine point L of $\triangle ABC$ is

$$l = 2\,\frac{\sigma_2\bar\sigma_1 - 3\,\sigma_1}{\sigma_1\bar\sigma_1 - 9}.$$

2°. The affix of the point that is isogonally conjugate to the orthocentre H with respect to the triangle ABC is

$$\frac{\sigma_1 + \bar\sigma_1\sigma_2 - \sigma_1 - \sigma_3\bar\sigma_1^2}{\sigma_1\bar\sigma_1 - 1} = \frac{\dfrac{\sigma_2^2}{\sigma_3} - \dfrac{\sigma_2^2}{\sigma_3}}{\sigma_1\bar\sigma_1 - 1} = 0,$$

that is, the orthocentre H and the centre O of the circle (ABC) are isogonally conjugate with respect to $\triangle ABC$ (the straight lines AO and AH are symmetric about the line AI and similarly BO and BH are symmetric about BI, CO and CH are symmetric about CI).

3°. The affix of the point isogonally conjugate to the centre O_9 of the Euler circle (O_9) with respect to $\triangle ABC$ is

$$\frac{\dfrac{\sigma_1}{2} + \dfrac{\bar\sigma_1}{2}\sigma_2 - \sigma_1 - \sigma_3\dfrac{\bar\sigma_1^2}{4}}{\dfrac{\sigma_1\bar\sigma_1}{4} - 1} = \frac{-\dfrac{\sigma_1}{2} + \dfrac{\sigma_2^2}{2\sigma_3} - \dfrac{\sigma_2^2}{4\sigma_3}}{\dfrac{\sigma_1\bar\sigma_1}{4} - 1}$$

$$= \frac{-\dfrac{\sigma_1}{2} + \dfrac{\sigma_2^2}{4\sigma_3}}{\dfrac{\sigma_1\bar\sigma_1}{4} - 1} = \frac{\sigma_2\bar\sigma_1 - 2\tau_1}{\sigma_1\bar\sigma_1 - 4}.$$

Problem 39. Prove that the midpoint M of a segment whose ends are points P_1 and Q_1, which are isogonal conjugates (with respect to $\triangle ABC$) of the ends P and Q of any diameter PQ of the circle (Ω), which is concentric with the circle (ABC), describes (as the diameter PQ rotates) a circle that touches the tangents drawn from the orthocentre H of $\triangle ABC$ to the circle (ABC). Triangle ABC is assumed to be obtuse, and the radius ρ of the circle (Ω) is assumed to be less than OH [only in this case is it possible to draw tangents from point H to (ABC)].

Solution. Take (ABC) for the unit circle; let z_1, z_2, z_3 be the respective affixes of the points A, B, C. Let PQ be an arbitrary diameter of (Ω), and let p and $-p$ be the respective affixes of the points P and Q.

Denoting by p_1 and q_1 the affixes of points P_1 and Q_1, which are the respective isogonal conjugates of the points P and Q with respect to $\triangle ABC$, we will have (see problem 37)

$$p_1 = \frac{\sigma_3\bar{p}^2 - \sigma_2\bar{p} - p + \sigma_1}{1 - p\bar{p}},$$

$$q_1 = \frac{\sigma_3\bar{p}^2 + \sigma_2\bar{p} + p + \sigma_1}{1 - p\bar{p}}.$$

From these we find the affix m of the midpoint M of segment P_1Q_1:

$$m = \frac{p_1 + q_1}{2} = \frac{\sigma_3\bar{p}^2 + \sigma_1}{1 - p\bar{p}}$$

From this relation it follows that if point P describes a circle (Ω) which is concentric with the circle (ABC), then $1 - p\bar{p}$=constant [$1 - p\bar{p} = -\sigma$, where σ is the power of the point P with respect to (ABC)] and, hence, point M describes a circle (T), the affix of whose centre is

$$t = \frac{\sigma_1}{1 - p\bar{p}}$$

and the radius is

$$R_1 = \frac{|\sigma_3\bar{p}^2|}{|1 - p\bar{p}|} = \frac{OP^2}{|1 - OP^2|}.$$

Knowing the affixes of the points H and T, we can find the complex number corresponding to the directed line segment $\overrightarrow{HT}$:

$$t - \sigma_1 = \frac{\sigma_1}{1 - p\bar{p}} - \sigma_1 = \frac{\sigma_1 p\bar{p}}{1 - p\bar{p}} = \sigma_1\frac{OP^2}{1 - OP^2}.$$

Associated with the directed line segment $\overrightarrow{HO}$ is the complex number $-\sigma_1$. Hence

$$\frac{\overrightarrow{HT}}{\overrightarrow{HO}} = \frac{OP^2}{OP^2 - 1}.$$

From this relation it follows that point T is the image of the centre O of the circle (ABC) under a homothetic transformation with centre H and ratio $\frac{OP^2}{OP^2 - 1}$. Therefore the circle (T) described by point M touches the tangents drawn to (ABC) from the orthocentre H of $\triangle ABC$ (only here is the condition used that $\triangle ABC$ is an obtuse-angled triangle).

Remarks. (1) The radius R_1 of (T) is equal to the radius 1 of (ABC) if and only if

$$\frac{OP^2}{|1 - OP^2|} = 1,$$

whence, obviously,

$$\frac{OP^2}{1 - OP^2} = 1, \quad OP = \frac{1}{\sqrt{2}} = \frac{R}{\sqrt{2}}.$$

(2) The radius R_1 of (T) is equal to the radius of the Euler circle if and only if

$$OP = \frac{R}{\sqrt{3}}.$$

(3) The radius R_1 of (T) is equal to the radius of (Ω) if and only if

$$OP = \frac{\sqrt{5} \pm 1}{2} R.$$

Problem 40. Given in a plane an arbitrary triangle ABC. The circle $(ABC) = (O)$ is taken as the unit circle. Let τ_0 be the affix of the centre I of the circle inscribed in $\triangle ABC$, and let τ_1, τ_2, τ_3 be the affixes of the centres I_a, I_b, I_c of the circles (I_a), (I_b), (I_c) escribed in the angles A, B, C of the given $\triangle ABC$. Prove that the orthopoles of the straight lines OI, OI_a, OI_b, OI_c with respect to $\triangle ABC$ are, respectively, the Feuerbach points Φ_0, Φ_1, Φ_2, Φ_3 of $\triangle ABC$: the points Φ_0, Φ_1, Φ_2, Φ_3 are, respectively, the points in which the Euler circle of $\triangle ABC$ is tangent to (I), (I_a), (I_b), (I_c). By proceeding from the foregoing, find the affixes φ_0, φ_1, φ_2, φ_3 of the Feuerbach points Φ_0, Φ_1, Φ_2, Φ_3. The affixes of the vertices A, B, C of $\triangle ABC$ are equal to z_1, z_2, z_3 respectively (see Fig. 35).

Solution. Let τ_0, τ_1, τ_2, τ_3 be the respective affixes of the points I, I_a, I_b, I_c. On the basis of problem 7, the orthopoles of the straight lines OI, OI_a, OI_b, OI_c with respect to $\triangle ABC$ have the affixes

$$\varphi_0' = \frac{1}{2}\left(\sigma_1 - \sigma_3 \frac{\bar{\tau}_0}{\tau_0}\right), \tag{142}$$

$$\varphi_1' = \frac{1}{2}\left(\sigma_1 - \sigma_3 \frac{\bar{\tau}_1}{\tau_1}\right), \tag{143}$$

$$\varphi_2' = \frac{1}{2}\left(\sigma_1 - \sigma_3 \frac{\bar{\tau}_2}{\tau_2}\right), \tag{144}$$

$$\varphi_3' = \frac{1}{2}\left(\sigma_1 - \sigma_3 \frac{\bar{\tau}_3}{\tau_3}\right) \tag{145}$$

since the slopes of the lines OI, OI_a, OI_b, OI_c are respectively,

$$\varkappa_0 = \frac{\tau_0}{\bar{\tau}_0}, \quad \varkappa_1 = \frac{\tau_1}{\bar{\tau}_1}, \quad \varkappa_2 = \frac{\tau_2}{\bar{\tau}_2}, \quad \varkappa_3 = \frac{\tau_3}{\bar{\tau}_3}.$$

We now have to prove that φ_0', φ_1', φ_2', φ_3' are the affixes of the respective Feuerbach points. Let us first consider the formula (142):

$$\varphi_0' = \frac{1}{2}\left(\sigma_1 - \sigma_3 \frac{\bar{\tau}_0}{\tau_0}\right).$$

Let o_9 be the affix of the centre of the circle (O_9) of $\triangle ABC$. Since $o_9 = \frac{\sigma_1}{2}$, it follows that

$$\varphi_0' - \frac{\sigma_1}{2} = \varphi_0' - o_9 = -\frac{\sigma_3\bar{\tau}_0}{2\tau_0},$$

whence

$$|\varphi_0' - o_9| = \frac{1}{2}$$

and so the point Φ_0' lies on the Euler circle. By the Euler formula for the distance between the centres of a circumscribed circle and an inscribed circle of the triangle,

$$OI^2 = R^2 - 2Rr = 1 - 2r \quad \text{(because we have } R = 1),$$

where r is the radius of (I) inscribed in $\triangle ABC$. Since $OI^2 = \tau_0\bar{\tau}_0$, it follows from the last formula that

$$r = \frac{1 - \tau_0\bar{\tau}_0}{2}.$$

The equation of the circle (I) is

$$(z - \tau_0)(\bar{z} - \bar{\tau}_0) = \frac{(1 - \tau_0\bar{\tau}_0)^2}{4}. \tag{146}$$

We now prove that the point Φ_0' with affix

$$\varphi_0' = \frac{1}{2}\left(\sigma_1 - \sigma_3 \frac{\bar{\tau}_0}{\tau_0}\right)$$

lies on the circle (I). We have

$$\varphi_0' - \tau_0 = \frac{\sigma_1}{2} - \frac{\sigma_3 \bar{\tau}_0}{2\tau_0} - \tau_0,$$

and since τ_0 is the fixed point of an isogonal transformation with respect to $\triangle\, ABC$ [see problem 37, formula (141)], it follows that

$$2\tau_0 + \sigma_3\, \bar{\tau}_0^2 - \sigma_1 = 0,$$

whence

$$\tau_0 = \frac{\sigma_1 - \sigma_3\, \bar{\tau}_0^2}{2}$$

and therefore

$$\varphi_0' - \tau_0 = \frac{\sigma_1}{2} - \frac{\sigma_3 \bar{\tau}_0}{2\tau_0} - \frac{\sigma_1}{2} + \frac{\sigma_3 \bar{\tau}_0^2}{2}$$

$$= \frac{\sigma_3 \bar{\tau}_0^2}{2\tau_0} - \frac{\sigma_3 \bar{\tau}_0}{2\tau_0} = \frac{\sigma_3 \bar{\tau}_0}{2\tau_0} (\tau_0 \bar{\tau}_0 - 1).$$

From this we get

$$\bar{\varphi}_0' - \bar{\tau}_0 = \frac{\bar{\sigma}_3 \tau_0}{2\bar{\tau}_0} (\tau_0 \bar{\tau}_0 - 1)$$

and, hence,

$$(\varphi_0' - \tau_0)(\bar{\varphi}_0' - \bar{\tau}_0) = \frac{(\tau_0 \bar{\tau}_0 - 1)^2}{4} = r^2.$$

That is, the point Φ_0' with affix φ' lies on the circle (I), the equation of which is of the form (146). Furthermore, the slope of the straight line $I\Phi_0'$ is

$$\frac{\varphi_0' - \tau_0}{\bar{\varphi}_0' - \bar{\tau}_0} = \sigma_3^2 \frac{\bar{\tau}_0^2}{\tau_0^2}$$

The slope of the straight line $O_9\Phi_0'$ is

$$\frac{\dfrac{\sigma_1}{2} - \varphi_0'}{\dfrac{\bar{\sigma}_1}{2} - \bar{\varphi}_0'} = \frac{\dfrac{1}{2} \sigma_3 \dfrac{\bar{\tau}_0}{\tau_0}}{\dfrac{1}{2} \bar{\sigma}_3 \dfrac{\tau_0}{\bar{\tau}_0}} = \sigma_3^2 \frac{\bar{\tau}_0^2}{\tau_0^2}$$

and this means the points O_9, Φ_0', I lie on one straight line. To summarize, then: the point Φ_0' with affix φ_0' lies both on the circle (I) and on the Euler circle (O_9), and, besides, the points Φ_0', I, O_9 lie on one straight line. Now if two circles have one common point (in this case, Φ_0') and their centres

(in the given case, I and O_9) lie on the same line as that common point, then they are tangent to one another at precisely that point. But the point of tangency of the circles (I) and (O_9) is precisely the Feuerbach point Φ_0, and so the points Φ_0' and Φ_0 coincide and, hence, $\varphi_0' = \varphi_0$.

To derive the other three formulas

$$\varphi_1' = \varphi_1, \quad \varphi_2' = \varphi_2, \quad \varphi_3' = \varphi_3,$$

where φ_1', φ_2', φ_3' are given by (143), (144), (145), and φ_1, φ_2, φ_3, are the affixes of the Feuerbach points Φ_1, Φ_2, Φ_3, we first derive the formulas

$$OI_a^2 = R^2 + 2r_aR, \tag{147}$$

$$OI_b^2 = R^2 + 2r_bR, \tag{148}$$

$$OI_c^2 = R^2 + 2\,r_cR \tag{149}$$

for the distances between the centres of the circumscribed circle and escribed circle (it is of course sufficient to prove only the first one of them). Let us consider the inversion $[I_a, r_a^2]$, whose centre is the centre I_a of the circle (I_a) escribed in the angle A of triangle ABC, and the power is equal to r_a. Let the circle (I_a) be tangent to the straight lines BC, CA, AB respectively at the points P_1, P_2, P_3 (see Fig. 35). Under the inversion $[I_a, r_a^2]$, the points A, B, C go respectively into the points A_1^*, B_1^*, C_1^*, in which the following straight lines intersect: I_aA and P_2P_3; I_aB and P_3P_1; I_aC and P_1P_2. The points A_1^*, B_1^*, C_1^* are the respective midpoints of the segments P_2P_3, P_3P_1, P_1P_2. Thus, the circles (ABC) and $(A_1^*B_1^*C_1^*)$ go into each other under this inversion with the circle of inversion $(P_1P_2P_3)=(I_a)$.The circle $(A_1^*B_1^*C_1^*)$ is the Euler circle of $\triangle\, P_1P_2P_3$ and therefore the radius of $(A_1^*B_1^*C_1^*)$ is equal to $R_a'=r_a/2$. On the other hand, the ratio R_a'/R of the radii of $(A_1^*B_1^*C_1^*)$ and (ABC) is equal to $|k/\sigma|$, where σ is the power of the centre I_a with respect to the circle (ABC), and k is the power of the inversion at hand, that is $k = r_a^2$. Indeed, suppose M and M' are corresponding points under the inversion at hand [point M on the circle (ABC) and point M' on the circle $(A_1^*B_1^*C_1^*)$]. Then

$$(I_aM)\,(I_aM') = r_a^2 \quad (= k). \tag{150}$$

Let M_1 be the second point of intersection of the line I_aM with the circle (ABC). Then

$$(I_aM)\,(I_aM_1) = \sigma. \tag{151}$$

From the relations (150) and (151) we find

$$\frac{(I_bM')}{(I_aM_1)} = \frac{k}{\sigma}, \quad (I_aM') = \frac{k}{\sigma}\,(I_aM_1).$$

Incidentally, this means that the circle $(O_1^*)=(A_1^*B_1^*C_1^*)$ may be obtained from the circle (ABC) by a homothetic transformation with the centre of similitude (homothetic centre) I_a and ratio k/σ. (Of course, the correspon-

dence of points under this homothetic transformation and under the inversion that we considered is different.) Since a homothetic transformation is a similarity transformation, it follows that

$$\frac{R_a'}{R} = \left|\frac{k}{\sigma}\right| \text{ or } \frac{\frac{1}{2} r_a}{R} = \frac{r_a^2}{|OI_a^2 - R^2|}.$$

Since the point I_a lies outside the circle (ABC) [the midpoint of segment II_a lies on the circle (ABC) — see problem 36], it follows that $OI_a > R$ and, hence,

$$\frac{r_a}{2R} = \frac{r_a^2}{OI_a^2 - R^2},$$

whence

$$OI_a^2 = R^2 + 2Rr_a.$$

We assumed $R=1$. Besides, $OI_a^2 = \tau_1\bar{\tau}_1$ since τ_1 is the affix of point I_a; hence, $r_a = \frac{1}{2}(\tau_1\bar{\tau}_1 - 1)$. We now prove that the point Φ_1' with affix

$$\varphi_1' = \frac{1}{2}\left(\sigma_1 - \sigma_3\frac{\bar{\tau}_1}{\tau_1}\right)$$

lies both on the Euler circle and on the circle (I_a). We have

$$\varphi_1' - \tau_1 = \frac{\sigma_1}{2} - \frac{\sigma_3\bar{\tau}_1}{2\tau_1} - \tau_1$$

and since I_a is a fixed point of an isogonal transformation with respect to $\triangle ABC$, it follows that

$$2\tau_1 + \sigma_3\bar{\tau}_1^2 - \sigma_1 = 0,$$

whence

$$\tau_1 = \frac{\sigma_1 - \sigma_3\bar{\tau}_1^2}{2}$$

and therefore

$$\varphi_1' - \tau_1 = \frac{\sigma_1}{2} - \frac{\sigma_3\bar{\tau}_1}{2\tau_1} - \frac{\sigma_1 - \sigma_3\bar{\tau}_1^2}{2}$$

$$= \frac{\sigma_3\bar{\tau}_1^2}{2} - \frac{\sigma_3\bar{\tau}_1}{2\tau_1} = \frac{\sigma_3\bar{\tau}_1}{2\tau_1}(\tau_1\bar{\tau}_1 - 1). \quad (152)$$

From this we have

$$\bar{\varphi}_1' - \bar{\tau}_1 = \frac{\bar{\sigma}_3\tau_1}{2\bar{\tau}_1}(\tau_1\bar{\tau}_1 - 1)$$

and so

$$(\varphi_1' - \tau_1)(\bar{\varphi}_1' - \bar{\tau}_1) = \frac{(\tau_1\bar{\tau}_1 - 1)^2}{4} = r_a^2.$$

That is, the point Φ_1' with affix φ_1' lies on the circle (I_a) whose equation is

$$(z - \tau_1)(\bar{z} - \bar{\tau}_1) = r_a^2.$$

Furthermore, $O_9 = \sigma_1/2$ is the affix of the centre of (O_9). Therefore,

$$\varphi_1' - o_9 = \frac{1}{2}\left(\sigma_1 - \sigma_3\frac{\bar{\tau}_1}{\tau_1}\right) - \frac{1}{2}\sigma_1 = -\frac{\sigma_3\bar{\tau}_1}{2\tau_1},$$

whence

$$|\varphi_1' - o_9| = 1/2$$

and therefore the point Φ_1' lies also on the Euler circle, the equation of which may be written down in the form $|z - o_9| = 1/2$. The slope of the straight line $I_a\Phi_1'$ is

$$\frac{\varphi_1' - \tau_1}{\bar{\varphi}_1' - \bar{\tau}_1} = \frac{\dfrac{\sigma_3\bar{\tau}_1}{2\tau_1}(\tau_1\bar{\tau}_1 - 1)}{\dfrac{\bar{\sigma}_3\tau_1}{2\bar{\tau}_1}(\tau_1\bar{\tau}_1 - 1)} = \sigma_3^2\frac{\bar{\tau}_1^2}{\tau_1^2}.$$

The slope of the straight line $O_9\Phi_1'$ is

$$\frac{\dfrac{\sigma_1}{2} - \varphi_1'}{\dfrac{\bar{\sigma}_1}{2} - \bar{\varphi}_1'} = \frac{\dfrac{1}{2}\sigma_3\dfrac{\bar{\tau}_1}{\tau_1}}{\dfrac{1}{2}\bar{\sigma}_3\dfrac{\tau_1}{\bar{\tau}_1}} = \sigma_3^2\frac{\bar{\tau}_1^2}{\tau_1^2}.$$

Consequently, the points O_9, I_a and Φ_1' lie on one straight line. Thus, the point Φ_1' also lies on the circle (O_9) and on the circle (I_a) and the centres O_9 and I_a of these circles lie on the same straight line as their common point Φ_1'; hence, (I_a) and (O_9) are tangent at the point Φ_1' with affix φ_1'. But the point of tangency of (I_a) and (O_9) is the Feuerbach point Φ_1, and so the points Φ_1' and Φ_1 coincide and $\varphi_1' = \varphi_1$.

In the same way we can prove that φ_2' and φ_3', that is, the affixes of the orthopoles of the straight lines OI_b and OI_c with respect to $\triangle ABC$ are the affixes of the Feuerbach points Φ_2 and Φ_3. That is, Φ_2' and Φ_3' are points of tangency of the circles (I_b) and (I_c) with the Euler circle (O_9) of $\triangle ABC$.

Problem 41. Let z_1, z_2, z_3 be the affixes of the points A, B, C in a system of coordinates in which $(ABC) = (O)$ is taken as the unit circle. It is required to prove that:

1°. The affixes τ_0, τ_1, τ_2, τ_3 of the centres of the circles inscribed and escribed in $\triangle ABC$ are given by the equation

$$\tau^4 - 2\sigma_2\tau^2 + 8\sigma_3\tau + \sigma_2^2 - 4\sigma_1\sigma_3 = 0.$$

2°. The affixes ψ_0, ψ_1, ψ_2, ψ_3 of the points Ψ_0, Ψ_1, Ψ_2, Ψ_3 symmetric to the orthocentre H of $\triangle ABC$ with respect to the Feuerbach points are found from the equation

$$(4\bar{\sigma}_2 - \bar{\sigma}_1^2)\psi^4 + 4\bar{\sigma}_1\psi^3 + 2\sigma_1\bar{\sigma}_1\psi^2 + 4\sigma_1\psi + 4\sigma_2 - \sigma_1^2 = 0.$$

These points Ψ_0, Ψ_1, Ψ_2, Ψ_3 lie on the circle (ABC) (see Fig. 42).

Solution. 1°. Let T be the centre of the circle inscribed in $\triangle ABC$ (or the centre of one of the escribed circles). Any one of these points has the *characteristic* property that it coincides with the point that is isogonally conjugate to it with respect to $\triangle ABC$. It is easy to verify the sufficiency of this condition geometrically; namely, any one of the indicated four points T is fixed under an isogonal transformation. We will now prove that there are four such points. Indeed, for point T with affix τ ($|\tau| \neq 1$) to coincide with its conjugate point, it is necessary and sufficient that the Morley relation

$$p + q + \sigma_3\bar{p}\bar{q} = \sigma_1$$

hold for $p = q = \tau$, that is,

$$2\tau + \sigma_3\bar{\tau}^2 = \sigma_1.$$

For example, we will prove that it holds for the affix

$$\tau_0 = -\sqrt{z_2}\sqrt{z_3} - \sqrt{z_3}\sqrt{z_1} - \sqrt{z_1}\sqrt{z_2}$$

of the centre of the circle (I) inscribed in $\triangle ABC$ (see problem 36). We have

$$\bar{\tau}_0 = -\frac{1}{\sqrt{z_2}\sqrt{z_3}} - \frac{1}{\sqrt{z_3}\sqrt{z_1}} - \frac{1}{\sqrt{z_1}\sqrt{z_2}} = -\frac{\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}}{\sqrt{z_1}\sqrt{z_2}\sqrt{z_3}},$$

or

$$\bar{\tau}_0^2 = \frac{1}{\sigma_3}[z_1 + z_2 + z_3 + 2(\sqrt{z_2}\sqrt{z_3} + \sqrt{z_3}\sqrt{z_1} + \sqrt{z_1}\sqrt{z_2})].$$

From this we have

$$2\tau_0 + \sigma_3\bar{\tau}_0^2 = 2(-\sqrt{z_2}\sqrt{z_3} - \sqrt{z_3}\sqrt{z_1} - \sqrt{z_1}\sqrt{z_2})$$
$$+ z_1 + z_2 + z_3 + 2(\sqrt{z_2}\sqrt{z_3} + \sqrt{z_3}\sqrt{z_1} + \sqrt{z_1}\sqrt{z_2}) = z_1 + z_2 + z_3 = \sigma_1,$$

and so also for the other affixes τ_1, τ_2, τ_3 [formulas (133) of problem 36].

Adjoining to the equation

$$2\tau + \sigma_3\bar{\tau}^2 - \sigma_1 = 0 \qquad (153)$$

the equation obtained by equating to zero the conjugate number of the left-hand side, we obtain

$$2\bar{\tau} + \bar{\sigma}_3\tau^2 - \bar{\sigma}_1 = 0, \qquad (154)$$

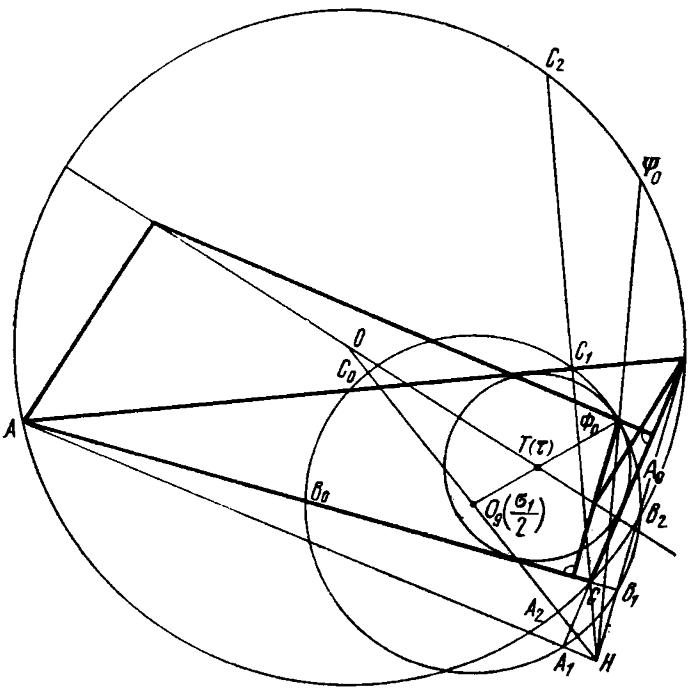

Fig. 42

whence

$$\bar{\tau} = \frac{\bar{\sigma}_1 - \tau^2 \bar{\sigma}^3}{2}$$

and the equation (153) takes the form

$$2\tau + \sigma_3 \frac{\bar{\sigma}_1^2 - 2\bar{\sigma}_1 \bar{\sigma}_3 \tau^2 + \tau^4 \bar{\sigma}_3^2}{4} - \sigma_1 = 0$$

or

$$8\tau + \frac{\sigma_2^2}{\sigma_3} - \frac{2\sigma_2}{\sigma_3}\tau^2 + \frac{1}{\sigma_3}\tau^4 - 4\sigma_1 = 0$$

or

$$\tau^4 - 2\sigma_2 \tau^2 + 8\sigma_3 \tau + \sigma_2^2 - 4\sigma_1 \sigma_3 = 0.$$

Since this is a fourth-degree equation, it follows that there cannot be more than four fixed points under an isogonal transformation. But the centre of the inscribed circle in $\triangle\, ABC$ and the centres of the circles escribed in this triangle are fixed under an isogonal transformation with respect to $\triangle\, ABC$. Hence, the fixed nature of a point under the isogonal transfor-

mation is the characteristic property of the centres of the circles (I), (I_a), (I_b), (I_c) (there are no other fixed points under an isogonal transformation with respect to $\triangle ABC$).

2°. The Feuerbach point Φ is an orthopole, with respect to $\triangle ABC$, of the straight line OI_k that passes through the centre O of the circle (ABC) and the centre I_k of the corresponding circle tangent to the lines BC, CA, AB. If the straight line is given by the equation

$$z - z_0 = \lambda(\bar{z} - \bar{z}_0),$$

then its orthopole with respect to $\triangle ABC$ has the affix

$$\mu = \frac{1}{2}\left(\sigma_1 - \frac{\sigma_3}{\lambda} + z_0 - \lambda\bar{z}_0\right)$$

(see problem 7). In particular, if the line at hand passes through the coordinate origin, then

$$\mu = \frac{1}{2}(\sigma_1 - \sigma_2\bar{\lambda}).$$

The affixes φ_k of the Feuerbach points Φ_k corresponding to the centres T_k of the four circles (T_k), each of which is tangent to the lines BC, CA, AB, will be (see problem 40)

$$\varphi_k = \frac{1}{2}\left(\sigma_1 - \sigma_3\frac{\bar{\tau}_k}{\tau_k}\right), \tag{155}$$

where τ_k is the affix of the point T_k (incidentally, it follows from this that the slope of line OT_k is equal to $\lambda_k = \frac{\tau_k}{\bar{\tau}_k}$ and so $\frac{\bar{\tau}_k}{\tau_k} = \bar{\lambda}_k$). We will denote the affixes τ_k and φ_k of the points T_k and Φ_k that correspond to each other by τ and φ. Then the relation (155) can be rewritten as

$$\varphi = \frac{1}{2}\left(\sigma_1 - \sigma_3\frac{\bar{\tau}}{\tau}\right). \tag{156}$$

Let us find the affix ψ of point Ψ, which is symmetric to point H with respect to point Φ:

$$\frac{\sigma_1 + \psi}{2} = \frac{1}{2}\left(\sigma_1 - \sigma_3\frac{\bar{\tau}}{\tau}\right),$$

whence

$$\psi = -\sigma_3\frac{\bar{\tau}}{\tau}. \tag{157}$$

Since $|\psi| = 1$, the point Ψ lies on the circle (ABC). Incidentally, this also follows immediately from the fact that under the homothetic transforma-

tion $(H, 2)$ the Euler circle of $\triangle ABC$ goes into the circle $(O) = (ABC)$, and since all Feuerbach points lie on the Euler circle, their images Ψ_0, Ψ_1, Ψ_2, Ψ_3 under the homothetic transformation $(H, 2)$ lie on the circle (ABC).

Let us eliminate τ and $\bar{\tau}$ from the equations (157), (153), (154). We obtain an equation for finding ψ. From the relation (157) we have

$$\bar{\tau} = -\psi\,\tau\,\bar{\sigma}_3$$

and, hence, the relations (153) and (154) take the form

$$2\tau + \sigma_3\,\psi^2\tau^2\bar{\sigma}_3^2 - \sigma_1 = 0 \tag{158}$$

$$-2\,\psi\,\tau\,\bar{\sigma}_3 + \bar{\sigma}_3\,\tau^2 - \bar{\sigma}_1 = 0 \tag{159}$$

or

$$\bar{\sigma}_3\,\psi^2\tau^2 + 2\,\tau - \sigma_1 = 0, \tag{160}$$

$$\bar{\sigma}_3\tau^2 - 2\,\psi\,\tau\,\bar{\sigma}_3 - \bar{\sigma}_1 = 0. \tag{161}$$

It remains to write the resultant of these equations as equal to zero. But this can be done differently: multiply Eq. (161) by $-\psi^2$ and add it to Eq. (160) to get

$$(2\psi^3\,\bar{\sigma}_3 + 2)\;\tau = \sigma_1 - \psi^2\bar{\sigma}_1,$$

whence

$$\tau = \frac{\sigma_1 - \psi^2\bar{\sigma}_1}{2\psi^3\,\bar{\sigma}_3 + 2}.$$

Substituting this value of τ, for example, into equation (161), we obtain

$$\bar{\sigma}_3\,\frac{\sigma_1^2 - 2\sigma_1\bar{\sigma}_1\psi^2 + \bar{\sigma}_1^2\psi^4}{4(\psi^3\bar{\sigma}_3 + 1)^2} - 2\psi\bar{\sigma}_3\,\frac{\sigma_1 - \psi^2\bar{\sigma}_1}{2(\psi^3\bar{\sigma}_3 + 1)} - \bar{\sigma}_1 = 0$$

or

$$(\bar{\sigma}_3\bar{\sigma}_1^2 - 4\sigma_1\bar{\sigma}_3^2)\,\psi^4 - 4\bar{\sigma}_1\bar{\sigma}_3\psi^3 - 2\sigma_1\bar{\sigma}_1\bar{\sigma}_3\psi^2 - 4\sigma_1\bar{\sigma}_3\psi + \bar{\sigma}_3\sigma_1^2 - 4\bar{\sigma}_1 = 0$$

or, cancelling $\bar{\sigma}_3$,

$$(\bar{\sigma}_1^2 - 4\sigma_1\bar{\sigma}_3)\psi^4 - 4\bar{\sigma}_1\psi^3 - 2\sigma_1\bar{\sigma}_1\psi^2 - 4\sigma_1\psi + \sigma_1^2 - 4\bar{\sigma}_1\sigma_3 = 0$$

or

$$(\bar{\sigma}_1^2 - 4\bar{\sigma}_2)\,\psi^4 - 4\bar{\sigma}_1\psi^3 - 2\,\sigma_1\bar{\sigma}_1\psi^2 - 4\sigma_1\psi + \sigma_1^2 - 4\sigma_2 = 0.$$

Note that the coefficients of this equation that are equidistant from the ends are conjugate in pairs: the first and the last, the second and the fourth (an antireciprocal equation).

Problem 42. Let M be a variable point of a circle (O) of radius R circumscribed about $\triangle ABC$; P is a point symmetric to the orthocentre H of $\triangle ABC$ with respect to the diameter of (O) parallel to the Simson line

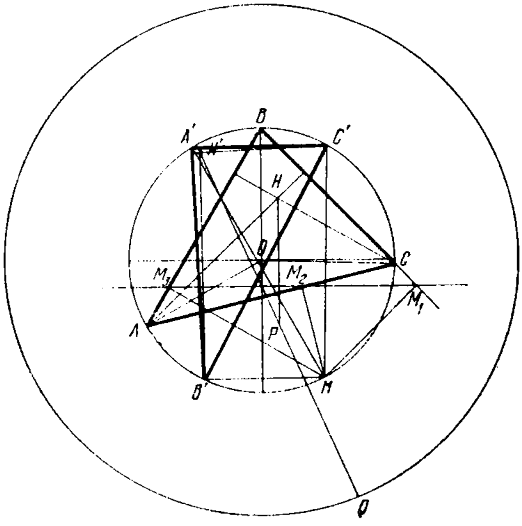

Fig. 43

for the point M with respect to $\triangle ABC$; A', B', C' are points symmetric to M about the straight lines OA, OB, OC. Prove that the point Q, which is symmetric with the orthocentre H' of $\triangle A'B'C'$ with respect to point P describes a circle concentric with (O) and having a radius OQ equal to

$$OQ = \frac{1}{R^3} OI \cdot OI_a \cdot OI_b \cdot OI_c,$$

where I is the centre of the circle inscribed in $\triangle ABC$, and I_a, I_b, I_c are the centres of the circles escribed in that triangle (Fig. 43).

Solution. Take (O) as the unit circle and let z_1, z_2, z_3, z_0 be the respective affixes of the points A, B, C, M. The equation of the straight line AB is

$$z + z_1 z_2 \bar{z} = z_1 + z_2. \tag{162}$$

The equation of the perpendicular dropped from point M to line AB is

$$z - z_0 = z_1 z_2 (\bar{z} - \bar{z}_0)$$

or

$$z - z_1 z_2 \bar{z} = z_0 - z_1 z_2 \bar{z}_0. \tag{163}$$

Adding the equations (162) and (163) termwise, we find the affix m_3 of the projection M_3 of point M on line AB:

$$m_3 = \frac{1}{2} (z_0 + z_1 + z_2 - z_1 z_2 \bar{z}_0).$$

Similarly, the affix m_2 of the projection M_2 of point M on line AC:

$$m_2 = \frac{1}{2}(z_0 + z_1 + z_3 - z_1 z_3 \bar{z}_0)$$

The slope of the straight line M_2M_3 is

$$\frac{m_3 - m_2}{\bar{m}_3 - \bar{m}_2} = \frac{z_2 - z_3 - z_1\bar{z}_0(z_2 - z_3)}{\bar{z}_2 - \bar{z}_3 - \bar{z}_1 z_0(\bar{z}_2 - \bar{z}_3)}$$

$$= \frac{(z_2 - z_3)(1 - z_1\bar{z}_0)}{(\bar{z}_2 - \bar{z}_3)(1 - \bar{z}_1 z_0)} = \frac{(z_2 - z_3)\dfrac{z_0 - z_1}{z_0}}{\dfrac{z_3 - z_2}{z_2 z_3} \cdot \dfrac{z_1 - z_0}{z_1}} = \frac{\sigma_3}{z_0},$$

and so the equation of the diameter of the circle parallel to the Simson line of point M with respect to $\triangle ABC$ is of the form

$$z = \frac{\sigma_3}{z_0}\bar{z} \tag{164}$$

or

$$z_0 z - \sigma_3 \bar{z} = 0. \tag{165}$$

Incidentally, this equation could have been written at once by proceeding from the equation of the Simson line and dropping the absolute term of the equation [problem 3, equation (1)].

The equation of the perpendicular dropped from point H to the straight line (165) is of the form

$$z - \sigma_1 = -\frac{\sigma_3}{z_0}(\bar{z} - \bar{\sigma}_1). \tag{166}$$

Adding the equations (164) and (166) term by term, we obtain

$$2z - \sigma_1 = \frac{\bar{\sigma}_1 \sigma_3}{z_0},$$

whence

$$z = \lambda = \frac{1}{2}\left(\sigma_1 + \frac{\sigma_2}{z_0}\right).$$

This is the affix of the projection of point H on the straight line (164). The affix p of point P is found from the relation

$$\frac{p + \sigma_1}{2} = \lambda = \frac{1}{2}\left(\sigma_1 + \frac{\sigma_2}{z_0}\right),$$

whence

$$p = \sigma_2 \bar{z}_0.$$

Furthermore, the equation of line OA is

$$z = z_1^2 \bar{z} \quad \text{or} \quad z - z_1^2 \bar{z} = 0.$$

The equation of the straight line passing through point M perpendicular to line OA is

$$z - z_0 = -z_1^2 (\bar{z} - \bar{z}_0).$$

Solving this equation together with the equation $z\bar{z} = 1$ of the unit circle (ABC), we obtain

$$z - z_0 = -z_1^2 \left(\frac{1}{z} - \frac{1}{z_0} \right),$$

$$z - z_0 = z_1^2 \frac{z - z_0}{z_0 z}.$$

One of the roots of this equation is naturally $z = z_0$ (the affix of point M) the other is

$$z = a' = z_1^2 \bar{z}_0,$$

which is the affix of point A'. In similar fashion we find the affixes b' and c' of points B' and C':

$$b' = z_2^2 \bar{z}_0, \qquad c' = z_3^2 \bar{z}_0,$$

and, hence, the affix h' of the orthocentre H' of $\triangle\ A'B'C'$:

$$h' = (z_1^2 + z_2^2 + z_3^2)\bar{z}_0.$$

The affix q of point Q, which is symmetric to point H' with respect to point P, is found from the relation

$$\frac{q + h'}{2} = p,$$

whence

$$\begin{aligned} q = 2p - h' &= 2\,\sigma_2 \bar{z}_0 - (z_1^2 + z_2^2 + z_3^2)\bar{z}_0 \\ &= \bar{z}_0 [2\sigma_2 - (z_1 + z_2 + z_3)^2 + 2(z_2 z_3 + z_3 z_1 + z_1 z_2)] = \bar{z}_0 (2\sigma_2 - \sigma_1^2 + 2\sigma_2) \\ &= \bar{z}_0 (4\sigma_2 - \sigma_1^2). \end{aligned}$$

Thus,

$$q = (4\sigma_2 - \sigma_1^2)\bar{z}_0.$$

From this it follows that if point M with affix z_0 describes a circle (ABC), then point Q describes a circle (Ω), in the opposite direction, that is con-

centric with (ABC). The radius ρ of (Ω) is

$$\rho = |4\sigma_2 - \sigma_1^2|.$$

The affixes of the points I, I_a, I_b, I_c are:

$$\tau_0 = -\sqrt{z_2}\sqrt{z_3} - \sqrt{z_3}\sqrt{z_1} - \sqrt{z_1}\sqrt{z_2} = a_0 + b_0 + c_0,$$
$$\tau_1 = -\sqrt{z_2}\sqrt{z_3} + \sqrt{z_3}\sqrt{z_1} + \sqrt{z_1}\sqrt{z_2} = a_0 - b_0 - c_0,$$
$$\tau_2 = \sqrt{z_2}\sqrt{z_3} - \sqrt{z_3}\sqrt{z_1} + \sqrt{z_1}\sqrt{z_2} = -a_0 + b_0 - c_0,$$
$$\tau_3 = \sqrt{z_2}\sqrt{z_3} + \sqrt{z_3}\sqrt{z_1} - \sqrt{z_1}\sqrt{z_2} = -a_0 - b_0 + c_0.$$

From this we have

$$\begin{aligned}
\tau_0\tau_1\tau_2\tau_3 &= [c_0^2 - (a_0 + b_0)^2]\,[c_0^2 - (a_0 - b_0)^2] \\
&= (c_0^2 - a_0^2 - b_0^2 - 2a_0b_0)\,(c_0^2 - a_0^2 - b_0^2 + 2a_0b_0) \\
&= (c_0^2 - a_0^2 - b_0^2)^2 - 4a_0^2b_0^2 = (a_0^2 + b_0^2 - c_0^2)^2 - 4a_0^2b_0^2 \\
&= (a_0^2 + b_0^2 + c_0^2)^2 - 4\,(b_0^2c_0^2 + c_0^2a_0^2 + a_0^2b_0^2) \\
&= (z_1z_2 + z_2z_3 + z_3z_1)^2 - 4\,(z_3^2z_1z_2 + z_1^2z_2z_3 + z_2^2z_3z_1) = \sigma_2^2 - 4\sigma_1\sigma_3
\end{aligned}$$

(see also problem 41). And so we get

$$\overline{\tau_0\tau_1\tau_2\tau_3} = \bar{\sigma}_2^2 - 4\bar{\sigma}_1\bar{\sigma}_3 = \frac{\sigma_1^2}{\sigma_2^2} - \frac{4\sigma_2}{\sigma_2^2} = \frac{\sigma_1^2 - 4\sigma_2}{\sigma_3^2}$$

and, thus,

$$|\overline{\tau_0\tau_1\tau_2\tau_3}| = |\tau_0\tau_1\tau_2\tau_3| = |\sigma_1^2 - 4\sigma_2| = \rho$$

or

$$\rho = |\tau_0|\,|\tau_1|\,|\tau_2|\,|\tau_3|$$

or

$$OQ = OI\cdot OI_a\cdot OI_b\cdot OI_c.$$

If the affixes of the points A, B, C are taken in the form Rz_1, Rz_2, Rz_3, where $|z_1| = |z_2| = |z_3| = 1$, then we obtain

$$OQ = \frac{OI\cdot OI_a\cdot OI_b\cdot OI_c}{R^3}.$$

Problem 43. The points A_1, A_2, A_3, A_4 lie on a single circle $(O) = (A_1A_2A_3A_4)$ which is taken as the unit circle. Denote by φ_1, φ_2, φ_3, φ_4 the affixes of the Feuerbach points Φ_1, Φ_2, Φ_3, Φ_4 that lie on the circles inscribed in the triangles $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_4$, $A_1A_2A_3$. Prove that the midpoints of the segments $A_1\Phi_1$, $A_2\Phi_2$, $A_3\Phi_3$. $A_4\Phi_4$ lie on one circle. Find the affix of the centre of this circle and its radius.

Solution. Let z_1, z_2, z_3, z_4 be the respective affixes of the points A_1, A_2, A_3, A_4. Then the affix φ_1 of the Feuerbach point Φ_1 of the circle inscribed in $\triangle A_2A_3A_4$ is computed from the formula (see problem 32, item 6°)

$$\varphi_1 = \frac{z_2z_3 + z_2z_4 + z_3z_4}{z_2 + z_3 + z_4}$$

and so the affix μ_1 of the midpoint of segment $A_1\Phi_1$ is

$$\mu_1 = \frac{\varphi_1 + z_1}{2} = \frac{z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4}{2(z_2 + z_3 + z_4)} = \frac{\sigma_2}{2(\sigma_1 - z_1)},$$

where

$$\sigma_2 = z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4,$$
$$\sigma_1 = z_1 + z_2 + z_3 + z_4.$$

We obtain similar expressions for the affixes μ_2, μ_3, μ_4 of the midpoints of the segments $A_2\Phi_2, A_3\Phi_3, A_4\Phi_4$:

$$\mu_2 = \frac{\sigma_2}{2(\sigma_1 - z_2)}, \quad \mu_3 = \frac{\sigma_2}{2(\sigma_1 - z_3)}, \quad \mu_4 = \frac{\sigma_2}{2(\sigma_1 - z_4)}.$$

From this it follows that the points μ_k $(k=1, 2, 3, 4)$ lie on the circle (Ω), the equation of which is

$$u = \frac{\sigma_2}{2\sigma_1 - 2z},$$

where z describes the unit circle.

Performing an inversion of this circle with the circle of inversion (O), we obtain the circle

$$v = \frac{1}{\bar{u}} = \frac{2\bar{\sigma}_1 - 2\bar{z}}{\bar{\sigma}_2} = 2\frac{\bar{\sigma}_1}{\bar{\sigma}_2} - \frac{2}{\bar{\sigma}_2}\bar{z}.$$

This is the circle (Ω'), the affix of whose centre is $2\bar{\sigma}_1/\bar{\sigma}_2$ and the radius is equal to $2/|\sigma_2|$. The circle (Ω) may be obtained from (Ω') by the same inversion and also by a similarity transformation. From this we can find the affix of the centre of (Ω) and its radius.

But we can also take advantage of the formulas of problem 12 that yield the affix of the centre,

$$\omega = \frac{b\bar{d} - a\bar{c}}{d\bar{d} - c\bar{c}},$$

and the radius

$$\rho = \left| \frac{bc - ad}{d\bar{d} - c\bar{c}} \right|$$

of (Ω) into which the unit circle (O) goes under the linear-fraction transformation

$$u = \frac{az + b}{cz + d}, \qquad ad - bc \neq 0. \tag{167}$$

Let us consider the transformation

$$u = \frac{\sigma_2}{2\sigma_1 - 2z}. \tag{168}$$

Here [compare with (167)], $a = 0$, $b = \sigma_2$, $c = -2$, $d = 2\sigma_1$ and so the affix ω of the centre of (Ω) on which the midpoints of the segments $A_1\Phi_1$, $A_2\Phi_2$, $A_3\Phi_3$, $A_4\Phi_4$ lie is

$$\omega = \frac{2\sigma_2\bar{\sigma}_1}{4(\sigma_1\bar{\sigma}_1 - 1)} = \frac{\sigma_2\bar{\sigma}_1}{2(\sigma_1\bar{\sigma}_1 - 1)}$$

and the radius is

$$\rho = \left|\frac{-2\sigma_2}{4\sigma_1\bar{\sigma}_1 - 4}\right| = \frac{1}{2}\left|\frac{\sigma_2}{\sigma_1\bar{\sigma}_1 - 1}\right|.$$

Problem 44. Let B_1 and C_1 be the points of intersection of the bisectors of the interior angles B and C of $\triangle ABC$ with the circle $(ABC) = (O)$. Consider the sum of the directed line segments $\overrightarrow{OB_1} + \overrightarrow{OC_1} = \overrightarrow{ON}$. Prove that $IN \perp BC$, where I is the centre of the circle inscribed in $\triangle ABC$. Also prove that the segment IN is equal to the radius of $(ABC) = (O)$.

Solution. Take (O) as the unit circle. Let z_1, z_2, z_3 be the respective affixes of the points A, B, C. The affixes a_1, b_1, c_1 of the points A_1, B_1, C_1 of intersection of the circle (ABC) and the bisectors of the interior angles of $\triangle ABC$ are given by the relations

$$a_1 = -\sqrt{z_2}\sqrt{z_3}, \quad b_1 = -\sqrt{z_3}\sqrt{z_1}, \quad c = -\sqrt{z_1}\sqrt{z_2},$$

where we take for the roots $\sqrt{z_1}, \sqrt{z_2}, \sqrt{z_3}$ values such that $|\sqrt{z_1}+\sqrt{z_2}+\sqrt{z_3}| < 1$. The affix n of point N is

$$n = -\sqrt{z_3}\sqrt{z_1} - \sqrt{z_1}\sqrt{z_2},$$

and since the affix of point I is equal to

$$\tau_0 = -\sqrt{z_2}\sqrt{z_3} - \sqrt{z_3}\sqrt{z_1} - \sqrt{z_1}\sqrt{z_2},$$

it follows that the directed line segment $\overrightarrow{IN}$ is associated with the complex number $\sqrt{z_2}\sqrt{z_3}$, whence $IN = |\sqrt{z_2}\sqrt{z_3}| = 1 = R$. The slope of the straight line IN is

$$\varkappa = \frac{\sqrt{z_2}\sqrt{z_3}}{\overline{\sqrt{z_2}}\,\overline{\sqrt{z_3}}} = z_2z_3.$$

The slope of line BC is equal to $\varkappa' = -z_2z_3$ and so $\varkappa + \varkappa' = 0$, and therefore the straight lines BC and IN are mutually perpendicular.

Problem 45. The angles of $\triangle ABC$ form a geometric progression with ratio 2. Prove that the midpoints of its sides and of the feet of the altitudes are six vertices of a regular heptagon (Fig. 44).

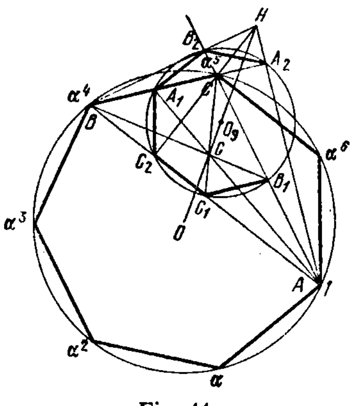

Fig. 44

Solution. Take (ABC) for the unit circle and assign to point A (the vertex of the smallest angle of the triangle) an affix of 1. Then, assuming

$$\alpha = \cos(2\pi/7) + i\sin(2\pi/7),$$

we conclude that $1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6$ are the affixes of the vertices of the regular heptagon inscribed in the circle (ABC). Under the conditions of the problem, the angles of the triangle are equal to A, $2A$ and $4A$, whence $A = \dfrac{2\pi}{7}$; therefore the affixes of the points B and C are equal, respectively, to α^4 and α^5. Let A_1, B_1, C_1 be the respective midpoints of the sides BC, CA, AB, and let a_1, b_1, c_1 be the affixes of these points. Then

$$a_1 = \frac{\alpha^4 + \alpha^5}{2}, \quad b_1 = \frac{\alpha^5 + 1}{2}, \quad c_1 = \frac{1 + \alpha^4}{2}.$$

The slope of line BC is equal to

$$\varkappa_{BC} = \frac{\alpha^4 - \alpha^5}{\bar{\alpha}^4 - \bar{\alpha}^5} = -\alpha^4\alpha^5 = -\alpha^9 = -\alpha^2$$

(since $\alpha^7 = 1$) and, hence, the equation of the straight line passing through point A perpendicularly to BC is

$$z - 1 = \alpha^2(\bar{z} - 1).$$

The equation of BC is

$$z - \alpha^4 = -\alpha^2(\bar{z} - \bar{\alpha}^4).$$

Solving the system of equations

$$z - 1 = \alpha^2(\bar{z} - 1)$$
$$z - \alpha^4 = -\alpha^2(\bar{z} - \bar{\alpha}^4),$$

we find

$$2z - 1 - \alpha^4 = -\alpha^2 + \bar{\alpha}^2 = -\alpha^2 + \alpha^5,$$

whence

$$z = a_2 = \frac{1 - \alpha^2 + \alpha^4 + \alpha^5}{2},$$

which is the affix of the foot A_2 of the altitude dropped from vertex A to side BC. Then

$$\varkappa_{AC} = \frac{1 - \alpha^5}{1 - \bar{\alpha}^5} = -\alpha^5.$$

The equation of AC is

$$z - 1 = -\alpha^5(\bar{z} - 1).$$

The equation of the altitude dropped from vertex B to side AC is

$$z - \alpha^4 = \alpha^5(\bar{z} - \bar{\alpha}^4).$$

From the last two equations we find

$$2z - 1 - \alpha^4 = \alpha^5 - \alpha,$$

whence

$$b_2 = \frac{1 - \alpha + \alpha^4 + \alpha^5}{2}.$$

And

$$\varkappa_{AB} = \frac{1 - \alpha^4}{1 - \bar{\alpha}^4} = -\alpha^4.$$

The equation of AB is

$$z - 1 = -\alpha^4(\bar{z} - 1).$$

The equation of the altitude dropped from vertex C to side AB is

$$z - \alpha^5 = \alpha^4(\bar{z} - \bar{\alpha}^5).$$

From the last two equations we find the affix c_2 of the projection of point C on line AB:

$$2c_2 - 1 - \alpha^5 = \alpha^4 - \bar{\alpha} = \alpha^4 - \alpha^6,$$

whence

$$c_2 = \frac{1 + \alpha^4 + \alpha^5 - \alpha^6}{2}.$$

Consider the affix o_9 of point O_9 obtained from point O under the homothetic transformation $\left(G, -\frac{1}{2}\right)$, that is, the affix of the centre O_9 of the Euler circle of $\triangle ABC$. Since the affix h of the orthocentre of $\triangle ABC$ is equal to $1 + \alpha^4 + \alpha^5$ and O_9 is the midpoint of OH, it follows that

$$o_9 = \frac{1 + \alpha^4 + \alpha^5}{2}.$$

We now find

$$a_2 - o_9 = \frac{1 - \alpha^2 + \alpha^4 + \alpha^5}{2} - \frac{1 + \alpha^4 + \alpha^5}{2} = -\frac{\alpha^2}{2},$$

$$b_2 - o_9 = \frac{1 - \alpha + \alpha^4 + \alpha^5}{2} - \frac{1 + \alpha^4 + \alpha^5}{2} = -\frac{\alpha}{2},$$

$$a_1 - o_9 = \frac{\alpha^4 + \alpha^5}{2} - \frac{1 + \alpha^4 + \alpha^5}{2} = -\frac{1}{2},$$

$$c_2 - o_9 = \frac{1 + \alpha^4 + \alpha^5 - \alpha^6}{2} - \frac{1 + \alpha^4 + \alpha^5}{2} = -\frac{\alpha^6}{2},$$

$$c_1 - o_9 = \frac{1 + \alpha^4}{2} - \frac{1 + \alpha^4 + \alpha^5}{2} = -\frac{\alpha^5}{2},$$

$$b_1 - o_9 = \frac{1 + \alpha^5}{2} - \frac{1 + \alpha^4 + \alpha^5}{2} = -\frac{\alpha^4}{2}.$$

These numbers form a geometric progression with ratio $1/\alpha$; and the moduli of all these differences are equal to $\frac{1}{2}$. This means that each of the directed line segments

$$\overrightarrow{O_9A_2},\ \overrightarrow{O_9B_2},\ \overrightarrow{O_9A_1},\ \overrightarrow{O_9C_2},\ \overrightarrow{O_9C_1},\ \overrightarrow{O_9B_1},$$

beginning with the second, is obtained from the preceding one by a rotation in one and the same direction through an angle of $2\pi/7$, that is, $A_2, B_2, A_1, C_2, C_1, B_1$ are the six successive vertices of a regular heptagon inscribed in the Euler circle of $\triangle ABC$.

Sec. 2. Problems with hints and answers

1. The angles A, B, C of a triangle are connected by the relation $C = 3B = 9A$; AA', BB', CC' are the altitudes of the triangle, H is its orthocentre, I is the centre of an inscribed circle; r is its radius, $(O) = (ABC)$ is a circle circumscribed about $\triangle ABC$; R is its radius, and a, b, c are the lengths of the sides BC, AC, AB. Prove that:

1°. $HA' + HB' - HC' = R/2$.

2°. $OI^2 + OH^2 = 5R^2$.

3°. $bc + ca + ab = R^2\sqrt{13}$.

4°. $\cos A + \cos B + \cos C = (1 + \sqrt{13})/4$.

5°. $OI^2 = R^2(5 - \sqrt{13})/2$.

6°. $r = R(\sqrt{13} - 3)/4$.

7°. $OH^2 = R^2(1 - 8\cos A\cos B\cos C) = R^2(5 + \sqrt{13})/2$.

8°. $\cos A \cos B \cos C = -(3+\sqrt{13})/16$.

9°. $\sin A \sin B \sin C = \sqrt{(13-3\sqrt{13})/128}$.

10°. $\sin A + \sin B + \sin C = \sqrt{(13+3\sqrt{13})/8}$.

Solution. The points A, B, C may be regarded as the respective vertices of a regular 26-gon inscribed in a circle of radius R, since $A=\pi/13$, $B=3\pi/13$, $C=9\pi/13$ (Fig. 45). First note that for any natural number,

$$\cos(\pi n/13) = \cos((26-n)\pi/13) = -\cos((13-n)\pi/13)$$

and, furthermore,

$$1+\cos\frac{2\pi}{13}+\cos\frac{4\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{8\pi}{13}+\cos\frac{10\pi}{13}+\cos\frac{12\pi}{13}$$
$$+\cos\frac{14\pi}{13}+\cos\frac{16\pi}{13}+\cos\frac{18\pi}{13}+\cos\frac{20\pi}{13}+\cos\frac{22\pi}{13}+\cos\frac{24\pi}{13}=0,$$

$$\cos\frac{2\pi}{13}+\cos\frac{4\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{8\pi}{13}+\cos\frac{10\pi}{13}+\cos\frac{12\pi}{13}$$
$$=\cos\frac{14\pi}{13}+\cos\frac{16\pi}{13}+\cos\frac{18\pi}{13}+\cos\frac{20\pi}{13}+\cos\frac{22\pi}{13}$$
$$+\cos\frac{24\pi}{13}=-1/2.$$

Set

$$\left.\begin{aligned} x &= \cos\frac{2\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{8\pi}{13},\\ y &= \cos\frac{4\pi}{13}+\cos\frac{10\pi}{13}+\cos\frac{12\pi}{13}.\end{aligned}\right\} \qquad (169)$$

Then

$$\cos A+\cos B+\cos C=-y.$$

Proceeding from the identities

$$2\cos^2\alpha = 1+\cos 2\alpha, \quad 2\cos\alpha\cos\beta = \cos(\alpha+\beta)+\cos(\alpha-\beta),$$

we obtain the equation

$$4x^2+2x-3=0,$$

whence

$$x=\frac{-1+\sqrt{13}}{4}, \quad y=\frac{-1-\sqrt{13}}{4},$$

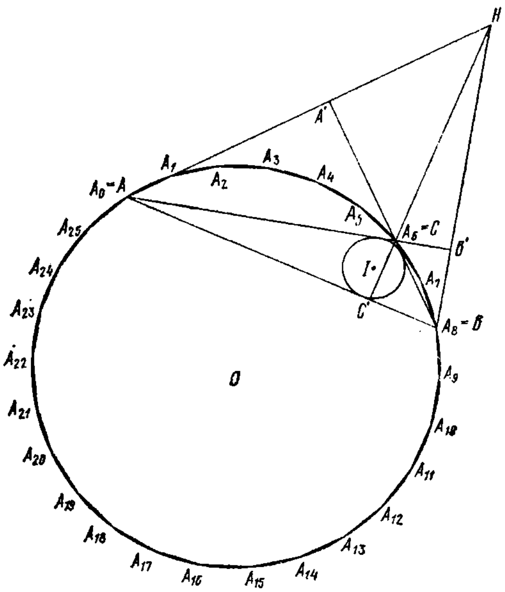

Fig. 45

and so

$$x + y = -\frac{1}{2}, \quad xy = -\frac{3}{4}.$$

This can also be obtained from (169).

1°. ABC is an obtuse-angled triangle ($C = 9\pi/13$), hence,

$$HA' = |2R \cos B \cos C| = -2R \cos B \cos C$$
$$= -R[\cos(B - C) + \cos(B + C)],$$
$$HB' = -R[\cos(C - A) + \cos(C + A)],$$
$$HC' = R[\cos(A - B) + \cos(A + B)].$$

From this we obtain

$$HA' + HB' - HC' = R(-x - y) = R/2.$$

2°. $OI^2 + OH^2 = R^2 - 2Rr + 9R^2 - a^2 - b^2 - c^2,$

$$2Rr = 8R^2 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$$
$$= 2R^2(-1 + \cos A + \cos B + \cos C) = 2R^2(-1 - y),$$

$$a^2 + b^2 + c^2 = 4R^2(\sin^2 A + \sin^2 B + \sin^2 C)$$
$$= 4R^2(3 - \cos^2 A - \cos^2 B - \cos^2 C) = 2R^2(3 - x),$$

and, thus,

$$OI^2 + OH^2 = R^2(6 + 2x + 2y) = 5R^2.$$

3°. $bc + ca + ab = 4R^2(\sin B \sin C + \sin C \sin A + \sin A \sin B)$

$$= 2R^2[\cos(B - C) - \cos(B + C) + \cos(C - A) - \cos(C + A)$$
$$+ \cos(A - B) - \cos(A + B)] = 2R^2(x - y) = R^2\sqrt{13}.$$

The derivation of the other relations is left to the reader. In deriving them, take advantage of the basic formulas relating to trigonometry and the geometry of any triangle; also take advantage of the specific nature of the given triangle ($A = \pi/13$, $B = 3\pi/13$, $C = 9\pi/13$).

It is also possible to introduce a complex system of coordinates by taking (ABC) as the unit circle. Then all the 26 values of $\sqrt[26]{1}$ will constitute the affixes of the vertices of the regular 26-gon mentioned earlier. These affixes are

$$a_k = \cos\frac{k\pi}{13} + i \sin\frac{k\pi}{13} \quad (k = 0, 1, 2, 3, \ldots, 23, 24, 25).$$

We can take point A as the unit point $a = a_0 = 1$. The points B and C will have the affixes

$$b = a_8 = \cos\frac{8\pi}{13} + i \sin\frac{8\pi}{13}, \quad c = a_6 = \cos\frac{6\pi}{13} + i \sin\frac{6\pi}{13}$$

(compare this with problem 45 above).

2. A quadrangle $ABCD$ is inscribed in a circle (O). Perpendiculars AA_2 and BB_2 are dropped on CD from points A and B; perpendiculars BB_1 and CC_1 are dropped on DA from points B and C; perpendiculars CC_2 and DD_2 are dropped on AB from points C and D; and, finally, from points D and A we drop perpendiculars DD_1 and AA_1 on BC.

Prove that (see Fig. 46):

1°. The segments A_1A_2, B_1B_2, C_1C_2, D_1D_2 are equal and lie on straight lines passing through point F, which is symmetric to point O with respect to the centroid of the system of points A, B, C, D.

2°. The quadrangles $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are similar to the quadrangle $ABCD$ and are inscribed in circles with centre F (I. Langer).

Hint. 1°. Take $(O) = (ABCD)$ for the unit circle. Let a, b, c, d be the respective affixes of the points A, B, C, D. The affix of point F is

$$f = \sigma_1/2 \quad (\sigma_1 = a + b + c + d).$$

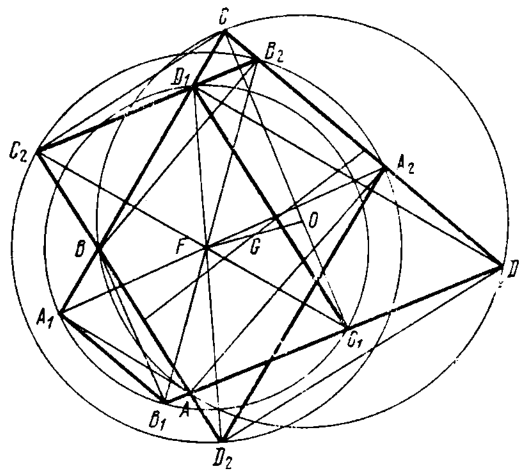

Fig. 46

The affixes of the projections A_1 and A_2 of point A on the lines BC and CD are

$$a_1 = \frac{1}{2}\left(a + b + c - \frac{bc}{a}\right), \quad a_2 = \frac{1}{2}\left(a + c + d - \frac{cd}{a}\right),$$

whence

$$a_1 - a_2 = \frac{1}{2a}(a - c)(b - d), \quad f - a_1 = \frac{1}{2a}(ad + bc).$$

Furthermore,

$$\frac{a_1 - a_2}{\bar{a}_1 - \bar{a}_2} = \frac{f - a_1}{\bar{f} - \bar{a}_1} = \frac{bcd}{a}$$

so that the line A_1A_2 passes through point F. Besides,

$$A_1A_2 = |a_1 - a_2| = \frac{1}{2} AC \cdot BD$$

and similarly for B_1B_2, C_1C_2, D_1D_2.

2°. We have

$$a_1 = f - \frac{1}{2a}(ad + bc), \quad a_2 = f - \frac{1}{2a}(ab + cd).$$

From this (and from similar relations for $b_1, b_2, c_1, c_2, d_1, d_2$) it follows that the quadrangles $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are inscribed in a circle with centre $F(f)$. Suppose the quadrangle $\overrightarrow{ABCD}$ is convex. The transfor-

mation $z' = f - \frac{1}{2}(ad+bc)\bar{z}$ carries quadrangle $\overrightarrow{ABCD}$ into the quadrangle $\overrightarrow{A_1B_1C_1D_1}$. The transformation $z'' = f - \frac{1}{2}(ab + cd)\bar{z}$ carries the quadrangle $\overrightarrow{ABCD}$ into the quadrangle $\overrightarrow{A_2B_2C_2D_2}$. And so the quadrangles $\overrightarrow{A_1B_1C_1D_1}$, $\overrightarrow{A_2B_2C_2D_2}$ are convex, have an orientation opposite that of quadrangle $\overrightarrow{ABCD}$, and all quadrangles $ABCD$, $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ are similar.

The radii of the circles in which the quadrangles $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are inscribed are equal, respectively, to

$$\frac{1}{2}|ad + bc| \quad \text{and} \quad \frac{1}{2}|ab + cd|$$

or

$$R|\cos(AD, BC)| \quad \text{and} \quad R|\cos(AB, CD)| \quad (R = 1).$$

Indeed, from the relations

$$a_1 = \frac{1}{2}\left(a + b + c - \frac{bc}{a}\right),$$

$$d_1 = \frac{1}{2}\left(b + c + d - \frac{bc}{d}\right)$$

we find

$$a_1 - d_1 = \frac{1}{2ad}(a - d)(ad + bc)$$

$$A_1D_1 = \frac{1}{2} AD\, |ad + bc|,$$

whence

$$\frac{1}{2}|ad + bc| = \frac{A_1D_1}{AD} = |(\cos(AD, BC)|.$$

Remark. The affix a_3 of the projection A_3 of point A on BD is

$$a_3 = f - \frac{1}{2a}(ac + bd).$$

Hence,

$$\frac{1}{3}(a_1 + a_2 + a_3) = f - \frac{\sigma_2}{6a},$$

where $\sigma_2 = ab + ac + ad + bc + bd + cd$. From this it follows that if the quadrangle Q (that is, $ABCD$) is inscribed in the circle (O), then the centroids of the projections of its vertices on the sides of the triangle, formed by the other three vertices of Q, form a quadrangle Q' similar to Q but with opposite orientation. Here, the quadrangle Q' is inscribed in a circle whose centre is symmetric to the point O with respect to the centroid of the quadrangle Q. The radius of this circle is equal to the distance from the centroid of the quadrangle Q to the centroid of the six points at which the circle (O) intersects the straight lines passing through some point of (O) parallel to the sides and diagonals of the quadrangle Q (the solution of Gourmagschieg).

3. Straight lines AP, BP, CP drawn from an arbitrary point P cut the sides BC, CA, AB of $\triangle ABC$ in points A_1, B_1, C_1 and divide these sides in the ratios:

$$\frac{\overrightarrow{BA_1}}{\overrightarrow{A_1C}} = \lambda, \quad \frac{\overrightarrow{CB_1}}{\overrightarrow{B_1A}} = \mu, \quad \frac{\overrightarrow{AC_1}}{\overrightarrow{C_1B}} = \nu.$$

Find the ratios

$$x = \frac{\overrightarrow{AP}}{\overrightarrow{PA_1}}, \quad y = \frac{\overrightarrow{BP}}{\overrightarrow{PB_1}}, \quad z = \frac{\overrightarrow{CP}}{\overrightarrow{PC_1}}.$$

Hint. Apply the Menelaus theorem to $\triangle AA_1B$ and to the transversal CC_1.

Answer. $x = (1 + \lambda)\,\nu$, $y = (1 + \mu)\,\lambda$, $z = (1 + \nu)\,\mu$. Complex numbers can also be used.

4. A triangle $A_1B_1C_1$ is inscribed in $\triangle ABC$ (that is, point A_1 lies on line BC, point B_1 on line CA and point C_1 on line AB).

1°. Prove that $\triangle ABC$ and $\triangle A_1B_1C_1$ have a common centroid if and only if the ratios

$$\lambda = \frac{\overrightarrow{BA_1}}{\overrightarrow{A_1C}}, \quad \mu = \frac{\overrightarrow{CB_1}}{\overrightarrow{B_1A}}, \quad \nu = \frac{\overrightarrow{AC_1}}{\overrightarrow{C_1B}}$$

are equal: $\lambda = \mu = \nu$.

2°. Assuming that $\triangle ABC$ and $\triangle A_1B_1C_1$ have a common centroid ($\lambda = \mu = \nu$), prove that

$$\frac{\overrightarrow{A_1B_1C_1}}{\overrightarrow{ABC}} = \frac{\lambda^2 - \lambda + 1}{(\lambda + 1)^2}.$$

3°. Suppose $\triangle A_2B_2C_2$ is circumscribed about $\triangle ABC$ and has a common centroid with $\triangle ABC$. Prove that the ratio λ', in which the points A, B, C

divide the directed line segments $\overrightarrow{B_2C_2}$, $\overrightarrow{C_2A_2}$, $\overrightarrow{A_2B_2}$, is then equal to $1/\lambda$, that is, the inverse (in magnitude) ratio in which the points A_1, B_1, C_1 divide $\overrightarrow{BC}, \overrightarrow{CA}, \overrightarrow{AB}$. Find

$$\frac{\overrightarrow{A_1B_1C_1}}{\overrightarrow{A_2B_2C_2}}$$

and prove that

$$(ABC)^2 = (A_1B_1C_1)(A_2B_2C_2)$$

(R. Deaux).

5. The triangles ABC and $A_1B_1C_1$ are homothetic under the homothetic transformation with centre S and ratio

$$k = \frac{SA}{SA_1}.$$

We know that $\triangle A'B'C'$ is inscribed in $\triangle ABC$ and circumscribed about $\triangle A_1B_1C_1$. Prove that we then have

$$(A'B'C')^2 = (ABC)(A_1B_1C_1)$$

(Pilatti's theorem).

6. Given a triangle ABC. Its sides are oriented as follows: $\overrightarrow{BC}, \overrightarrow{CA}, \overrightarrow{AB}$. Prove that on the axes $\overrightarrow{BC}, \overrightarrow{CA}, \overrightarrow{AB}$ there is only one triplet of points A_1, B_1, C_1 such that $(BA_1) = (CB_1) = (AC_1)$ and the straight lines AA_1, BB_1, CC_1 intersect in one point. Also prove that the points A_1, B_1, C_1 are the respective interior points of the segments BC, CA, AB.

Hint. Use the Ceva theorem. Investigate the resulting cubic equation $[(BA_1) = (CB_1) = (AC_1) = x]$ for x (S. Vatricant).

7. Let I, I_a, I_b, I_c be the centre of the circle (I) inscribed in $\triangle ABC$, and the centres of the circles $(I_a), (I_b), (I_c)$ escribed in the triangle ABC. Let R be the radius of the circle $(O) = (ABC)$, and let N be a point located on segment HH_1 at a distance of $(2/3)\, HH_1$ from H; H is the orthocentre of $\triangle ABC$, and H_1 is the orthocentre of $\triangle A_1B_1C_1$, whose vertices are the feet of the altitudes of $\triangle ABC$. Denote by L the centroid of a system of four points obtained from the points I, I_a, I_b, I_c by inversion with respect to the circle (O). Prove that the points O, N, L lie on one straight line and that

$$OI\cdot OI_a\cdot OI_b\cdot OI_c = 3R^2\cdot ON,$$

$$ON\cdot OL = \frac{2}{3} R^2.$$

8. A triangle ABC is inscribed in a circle (O). Construct right-angled triangles $\overrightarrow{AOA'}, \overrightarrow{BOB'}, \overrightarrow{COC'}$ (in all three triangles, the angle O is equal

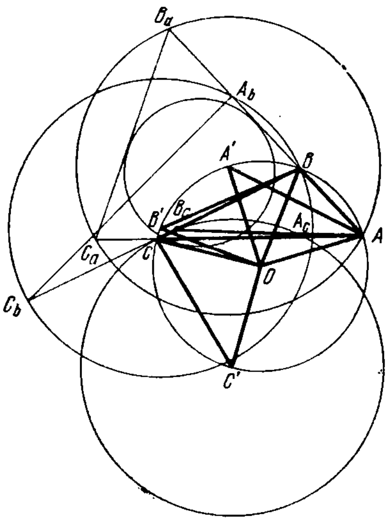

Fig. 47

to $\pi/2$) with the same orientation. Let the circles $(A', A'A)$, $(B', B'B)$, $(C', C'C)$ cut the sides AB and AC, BC and BA, CA and CB in the points B_a, C_a; C_b, A_b; A_c, B_c.

Prove that the triangles AB_aC_a, BC_bA_b, CA_cB_c have the same Euler circle (Fig. 47) (R. Blanchard).

Hint. Take (O) as the unit circle. Let a, b, c, a', b', c' be the affixes of the points A, B, C, A', B', C'. Then

$$a' = ia, \quad b' = ib, \quad c' = ic, \quad AA' = \sqrt{2}.$$

The equation of the circle $(A', A'A)$ is

$$(z - ia)(\bar{z} + i\bar{a}) = 2. \tag{170}$$

The equation of AB is

$$z + ab\bar{z} = a + b,$$

whence

$$\bar{z} + \bar{a}\bar{b}z = \bar{a} + \bar{b},$$

$$\bar{z} = \bar{a} + \bar{b} - \bar{a}\bar{b}z.$$

and equation (170) takes the form

$$(z - ia)(\bar{a} + \bar{b} - \bar{a}\bar{b}z + i\bar{a}) = 2$$

or

$$(z - ia)(\bar{a}\bar{b}z - \bar{a} - \bar{b} - i\bar{a}) + 2 = 0.$$

One of the roots of this equation is $z = a$, the other is

$$b_a = b + i(a + b).$$

Similarly, the affix c_a of point C_a is

$$c_a = c + i(a + c).$$

The affix h_a of the orthocentre H_a of $\triangle AB_aC_a$ is found from the relation

$$\overrightarrow{OH_a} = \overrightarrow{OA'} + \overrightarrow{A'B_a} + \overrightarrow{A'C_a} + \overrightarrow{A'A}.$$

That is,

$$h_a = ia + b + i(a + b) - ia + c + i(a + c) - ia + a - ia = \sigma_1(1+i) - ai.$$

The affix a_9 of the midpoint of H_aA', that is, of the centre of the Euler circle of $\triangle AB_aC_a$, is

$$a_9 = \frac{\sigma_1(1 + i) - ai + ai}{2} = \frac{1 + i}{2}\sigma_1 \, (= b_9 = c_9).$$

9. Given the affixes z_1, z_2, z_3 of the vertices A, B, C of $\triangle ABC$. Find- the affix z_0 of the centre of the circle (ABC).

Answer: $$z_0 = \frac{z_1\bar{z}_1(z_3 - z_2) + z_2\bar{z}_2(z_1 - z_3) + z_3\bar{z}_3(z_2 - z_1)}{\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}}$$

Hint. $(z_0 - z_1)(\bar{z}_0 - \bar{z}_1) = (z_0 - z_2)(\bar{z}_0 - \bar{z}_2) = (z_0 - z_3)(\bar{z}_0 - \bar{z}_3)$.

10. Let A_1, B_1, C_1 be points symmetric to the centre O of $(O) = (ABC)$ circumscribed about $\triangle ABC$ with respect to the sides BC, CA, AB respectively. Prove that $\triangle ABC$ and $\triangle A_1B_1C_1$ are symmetric about the midpoint O_9 of segment OH, where H is the orthocentre of $\triangle ABC$ *(Hamilton's theorem).*

Hint. Take (ABC) for the unit circle. Let z_1, z_2, z_3 be the affixes of point A, B, C. Then the affixes of the points A_1, B_1, C_1 will be $z_2 + z_3$, $z_3 + z_1$, $z_1 + z_2$. The affixes of the midpoints of segments AA_1, BB_1 and CC_1 are equal to $(z_1 + z_2 + z_3)/2 = \sigma_1/2$, and this is the affix of the midpoint of segment OH.

11. Given in the plane a similarity transformation consisting of a rotation about a point O through angle $\alpha \neq \pi n$ (n an integer) and of homothetic transformations with centre O and ratio k. Under this similarity transformation, let $\triangle ABC$ go into $\triangle A'B'C'$. Denote by A_1, B_1, C_1 the respective points of intersection of the straight lines BC and $B'C'$, CA and $C'A'$, AB and $A'B'$. What must the position of O be with respect to $\triangle ABC$ so that $\triangle\overrightarrow{ABC}$ and $\triangle\overrightarrow{A_1B_1C_1}$ are similar and have the same orientation?

Hint. Let a, b, c be the affixes of the points A, B, C in an arbitrary system of coordinates with origin at point O. Setting $p = k(\cos\alpha + i\sin\alpha)$, we find the affixes $a' = pa$, $b' = pb$, $c' = pc$ of points A', B', C'. From the equations of the lines BC and $B'C'$,

$$(\bar{b} - \bar{c})\,z - (b - c)\,\bar{z} + b\bar{c} - \bar{b}c = 0,$$

$$\bar{p}(\bar{b} - \bar{c})\,z - p(b - c)\,\bar{z} + p\bar{p}(b\bar{c} - \bar{b}c) = 0,$$

we find the affix $z = a_1$ of point A_1

$$a_1 = \frac{b\bar{c} - \bar{b}c}{b - c}\,\frac{p(\bar{p} - 1)}{p - \bar{p}},$$

and similar expressions for b_1 and c_1:

$$b_1 = \frac{c\bar{a} - \bar{c}a}{\bar{c} - \bar{a}}\,\frac{p(\bar{p} - 1)}{p - \bar{p}}, \qquad c_1 = \frac{a\bar{b} - \bar{a}b}{\bar{a} - \bar{b}}\,\frac{p(\bar{p} - 1)}{p - \bar{p}}.$$

The oriented triangles $\overrightarrow{ABC}$ and $\overrightarrow{A_1B_1C_1}$ are similar and have the same orientation if and only if

$$\begin{vmatrix} a & a_1 & 1 \\ b & b_1 & 1 \\ c & c_1 & 1 \end{vmatrix} = 0.$$

Substituting into this equation the expressions for a_1, b_1, c_1 in terms of a, b, c, p and simplifying, we obtain

$$a\bar{a}(\bar{b} - \bar{c}) + b\bar{b}(\bar{c} - \bar{a}) + c\bar{c}(\bar{a} - \bar{b}) = 0,$$

and this means (see the preceding problem) that O is the centre of the circle (ABC).

Remark. If we now take $(O) = (ABC)$ for the unit circle, then

$$a_1 = \frac{p(\bar{p} - 1)}{\bar{p} - p}\,(b + c),$$

$$b_1 = \frac{p(\bar{p} - 1)}{\bar{p} - p}\,(c + a),$$

$$c_1 = \frac{p(\bar{p} - 1)}{\bar{p} - p}\,(a + b),$$

and, hence, $\triangle\overrightarrow{A_1B_1C_1}$ is the image of $\triangle\overrightarrow{A_2B_2C_2}$ (which "supplements" $\triangle\overrightarrow{ABC}$) under the similarity transformation with centre O, angle of rotation

$$\arg\frac{p(\bar{p} - 1)}{\bar{p} - p}$$

and proportionality factor

$$2\left|\frac{p(\bar{p}-1)}{\bar{p}-p}\right| = \frac{\sqrt{k^2 - 2k\cos\alpha + 1}}{|\sin\alpha|},$$

where $k = |p|$ (R. Blanchard).

12. Prove that if in a triangle ABC the median m_a issuing from angle A is equal to the length l_a of the bisector of the same angle A, $m_a = l_a$, then $\triangle ABC$ is an isosceles triangle and $b = c$. Solve the problem analytically (by computation) and geometrically.

13. $A_1A_2A_3A_4A_5A_6A_7$ is a regular convex heptagon. Prove that the circle circumscribed about the triangle formed by the lines A_1A_2, A_3A_5, A_4A_7 passes through the vertex A_6 (V. Thebault).

14. Prove that if the angles of $\triangle ABC$ are

$$A = \frac{\pi}{7}, \quad B = \frac{2\pi}{7}, \quad C = \frac{4\pi}{7}$$

and if a, b, c denote the lengths of the sides BC, CA, AB, then the following relations hold true:

1°. $a^2 + b^2 + c^2 = 7R^2$.

2°. $OH = R\sqrt{2}$ (see problem 11).

3°. The angles of $\triangle A'B'C'$, whose vertices are the feet A', B', C' of the altitudes of the given triangle, are

$$A' = \frac{2\pi}{7} = B, \quad B' = \frac{4\pi}{7} = C, \quad C' = \frac{\pi}{7} = A.$$

4°. $\cos A = \dfrac{b}{2a}$, $\cos B = \dfrac{c}{2b}$, $\cos C = \dfrac{a-b}{2b} = -\dfrac{a}{2c}$.

5°. $bc = a(b+c)$, $bc = c^2 - a^2$, $ac = b^2 - a^2$ and, conversely, if the first two relations hold, then $A = \pi/7$, $B = 2\pi/7$, $C = 4\pi/7$.

6°. $\cos A \cos B \cos C = -1/8$, $\sin A \sin B \sin C = \sqrt{7}/8$.

7°. $\sin^2 A = \dfrac{3a-c}{4a}$, $\sin^2 B = \dfrac{3b-a}{4b}$, $\sin^2 C = \dfrac{3c+b}{4c}$.

8°. $\sin^2 A \sin^2 B \sin^2 C + \cos^2 A \cos^2 B \cos^2 C = 1/8$.

9°. $\cos^2 A + \cos^2 B + \cos^2 C = 5/4$.

10.° $\sin^2 A + \sin^2 B + \sin^2 C = 7/4$.

11°. $\cos A + \cos B + \cos C = \dfrac{b}{a} - \dfrac{1}{2} = \dfrac{c+a}{2(c-a)}$.

12°. $\cos^2 B \cos^2 C + \cos^2 C \cos^2 A + \cos^2 A \cos^2 B = 3/8$.

13°. $\tan A \tan B \tan C = -\sqrt{7}$.

14°. $h_a = h_b + h_c$.

15°. $h_a^2 + h_b^2 + h_c^2 = (a^2 + b^2 + c^2)/2$.

16°. $BA' \cdot A'C = \dfrac{ac}{4}$, $CB' \cdot B'A = \dfrac{ab}{4}$, $AB' \cdot C'B = \dfrac{bc}{4}$.

17°. If Q_a, Q_b, Q_c are the feet of the bisectors of the interior angles of $\triangle ABC$, and Q'_a, Q'_b, Q'_c are the feet of the bisectors of the exterior angles of the triangle, then

$$AQ_a = \sqrt{\frac{b^2 a^2 - a^4}{bc}}, \quad BQ_b = \frac{bc}{a+c} = c - a, \quad CQ_c = \frac{ac}{a+b} = b - a.$$

18°. $AQ_a^2 + BQ_b^2 + CQ_c^2 = 6a^2$.
19°. $BQ_b \cdot CQ_c = (c-a)(b-a) = a^2$.
20°. $AQ'_a = b + c$.
21°. $BQ_b'^2 = a^2 - ab + b^2$.
22°. $CQ_c'^2 = b^2 + bc + c^2$.
23°. $AQ_a'^2 + BQ_b'^2 + CQ_c'^2 = 4b^2 + 4c^2 - 2a^2 = 8m_a^2$.
24°. The radii of circles of Apollonius are equal to the sides of $\triangle ABC$ (the circles of Apollonius of the triangle are $(Q_aQ'_a), (Q_bQ'_b), (Q_cQ'_c)$ constructed on the segments $Q_aQ'_a, Q_bQ'_b, Q_cQ'_c$ as diameters).

Let $A_tB_tC_t = T_t$ be the triangle formed by tangents at points A, B, C to the circle (ABC) circumscribed about $\triangle ABC = T$. Prove that:

25°. The triangles $\overrightarrow{ABC}$ and $\overrightarrow{A_tB_tC_t}$ are similar and have opposite orientations; the proportionality factor is equal to 1/2.

26°. The altitudes AA', BB', CC' of $\triangle ABC$ bisect the sides B_tC_t, C_tA_t, A_tB_t of $\triangle T_t$.

27°. The Euler circle of $\triangle T_t$ passes through the orthocentre of $\triangle T$.

28°. The triangles $\overrightarrow{II_bI_c}$ and $\overrightarrow{ABC}$ are similar, have the same orientation, and the proportionality factor is equal to 1/2.

15. Given a triangle ABC, G is the point of intersection of the medians. Let A_1, B_1, C_1 be the images of the points A, B, C under the homothetic transformation $(G, -2)$. Let

$$\overrightarrow{BCA'},\ \overrightarrow{CAB'},\ \overrightarrow{ABC'} \tag{171}$$

be equilateral triangles having the same orientation; let

$$\overrightarrow{BCA''},\quad \overrightarrow{CAB''},\quad \overrightarrow{ABC''} \tag{172}$$

also be equilateral triangles having the same orientation; note that the orientation of any one of the triangles (171) is opposite that of any one of the triangles (172). Prove that the triangles

$$\overrightarrow{B'C'A_1},\quad \overrightarrow{C'A'B_1},\quad \overrightarrow{A'B'C_1} \tag{173}$$

are equilateral and have the same orientation as that of the triangles (172), and the triangles

$$\overrightarrow{B''C''A_1},\quad \overrightarrow{C''A''B_1},\quad \overrightarrow{A''B''C_1} \tag{174}$$

are equilateral and have the same orientation as that of the triangles (171) (Fig. 48).

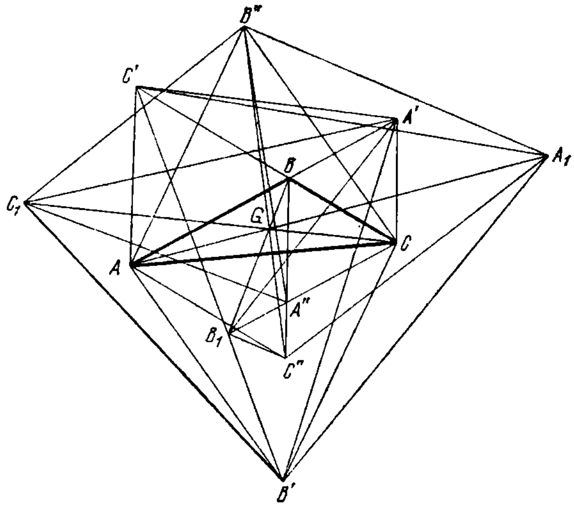

Fig. 48

Hint. If α is any one of the imaginary values of $\sqrt[3]{-1}$, then the affixes of the points $A', B', C', A'', B'', C''$ are expressed in terms of the affixes of the points A, B, C by the relations

$$a' = \alpha c + \bar{\alpha} b,$$

$$b' = \alpha a + \bar{\alpha} c,$$

$$c' = \alpha b + \bar{\alpha} a;$$

$$a'' = \bar{\alpha} c + \alpha b,$$

$$b'' = \bar{\alpha} a + \alpha c,$$

$$c'' = \bar{\alpha} b + \alpha a.$$

The affixes of the third vertices of the equilateral triangles constructed on the segments $B'C'$ and $B''C''$ and with orientation the same as $\triangle \overrightarrow{BCA''}$ and $\triangle \overrightarrow{BCA'}$, respectively, are

$$\bar{\alpha} c' + \alpha b' = b + c - a$$

$$\alpha c'' + \bar{\alpha} b'' = b + c - a$$

and, hence, these third vertices coincide with the point A_1 (R. Deaux).

16. Prove that if the angles A, B, C in $\triangle ABC$ are connected by the relation

$$\sin A = \cos B \tan C,$$

then the altitude AA', the median BB', and the bisector CC' intersect in one point.

Hint. The angles B and C are acute. Hence, the foot A' of the altitude AA' is an interior point of segment BC and

$$\frac{\overrightarrow{BA'}}{\overrightarrow{A'C}} = \frac{c \cos B}{b \cos C}.$$

From there on use Ceva's theorem.

Supplementary question. What theorem results from what has been proved if we apply an isogonal transformation to point M in which the altitude AA', the median BB', and the bisector CC' intersect? (after Morley).

Hint. Since, under an isogonal transformation, the centre O of (ABC) goes into the orthocentre H of $\triangle ABC$ (and conversely), we get the following theorem: if angles A, B, C in $\triangle ABC$ are connected by the relation $\sin A = \cos B \tan C$, then the diameter AO of the circle $(O) = (ABC)$, the bisector CC', and the cimedian BB^* (the straight line symmetric to the median BB' with respect to the bisector of the interior angle B) intersect in one point.

17. Inscribed in the unit circle is a triangle ABC, the affixes of whose vertices are respectively equal to z_1, z_2, z_3. Given a point P whose affix is

$$p = \frac{\alpha z_1 + \beta z_2 + \gamma z_3}{\alpha + \beta + \gamma},$$

where α, β, γ are real numbers. Let A_1, B_1, C_1 be points in which the straight lines AP, BP, CP intersect the sides BC, CA, AB respectively. Find

$$\frac{(A_1B_1C_1)}{(ABC)}.$$

Hint. The affixes of the points A_1, B_1, C_1 are

$$a_1 = \frac{\beta z_2 + \gamma z_3}{\beta + \gamma}, \quad b_1 = \frac{\gamma z_3 + \alpha z_1}{\gamma + \alpha}, \quad c_1 = \frac{\alpha z_1 + \beta z_2}{\alpha + \beta}.$$

Answer. $\dfrac{(A_1B_1C_1)}{(ABC)} = \dfrac{2\alpha\beta\gamma}{(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)}.$

18. Prove that a necessary and sufficient condition for the orthogonal projections B_1 and C_1 of the vertices B and C of $\triangle ABC$ on the sides AC and AB respectively to lie on the same straight line as the centroid G of $\triangle ABC$ is the equality $3a^2 = b^2 + c^2$ or $a^2 = b^2 + c^2$, where a, b, c are the lengths of the sides BC, CA, AB of $\triangle ABC$.

Hint. Take (ABC) as the unit circle. Prove the equations $z_2\bar{z}_3 + z_3\bar{z}_2 = 2 - a^2$ and so forth, where z_1, z_2, z_3 are the affixes of the vertices of the triangle, and a, b, c are the lengths of its sides.

19. Let z_1, z_2, z_3 be the affixes of the vertices of $\triangle ABC$ inscribed in the unit circle. Prove that the equation $z_1^2 = z_2 z_3$ is a necessary and sufficient condition for $\triangle ABC$ to be an isosceles triangle: $AC = AB$.

20. ABC is a right-angled triangle ($C = \pi/2$); (I) is a circle inscribed in the triangle. Let A_1, B_1, C_1 be the orthocentres of the triangles IBC, ICA, IAB. Prove that the length of the projection $A_3^*B_3^*$ of segment A_1B_1 on the hypotenuse AB is equal to the diameter of (I). The length of the projection $B_1^*C_1^*$ of segment B_1C_1 on leg BC is equal to the diameter of the circle (I_b) escribed in angle B, and the length $A_2^*C_2^*$, the projection of segment A_1C_1 on leg AC, is equal to the diameter of the circle (I_a) escribed in angle A.

Hint. Take (I) for the unit circle. Let P, Q, R be the points of tangency of (I) with the sides BC, CA, AB, and let their affixes be $1, i, \alpha$ respectively. Then the affixes of the points A, B, C are

$$a = \frac{2}{\bar{\alpha} + i} = \frac{2\alpha i}{\alpha + i}, \quad b = \frac{2}{1 + \bar{\alpha}} = \frac{2\alpha}{1 + \alpha},$$

$$c = \frac{2}{1 + \bar{i}} = 1 + i$$

since the points A, B, C are images, under inversion with respect to circle (I), of the points A', B', C' (the points of intersection of the straight lines IA, IB, IC with lines RQ, PR, PQ, respectively) (Fig. 49). Now, the affix a_1 of the orthocentre A_1 of $\triangle IBC$ is $a_1 = \dfrac{(1 - i)(i + \alpha)}{1 + \alpha}$ and so forth the calculations are rather unwieldy).

21. ABC is a right-angled triangle ($C = \pi/2$). The circle $(ABC) = (O)$ is taken as the unit circle. The point $M(z_0)$ describes this circle. Let Q

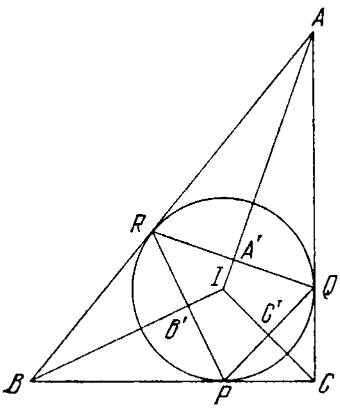

Fig. 49

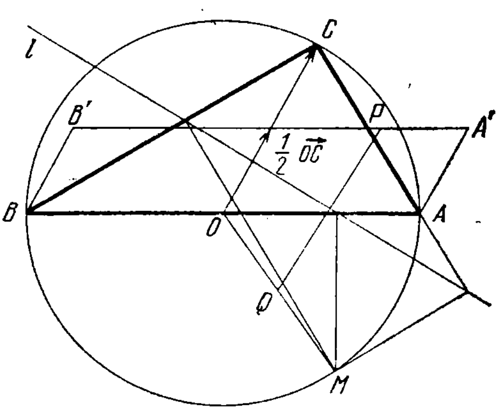

Fig. 50

be the midpoint of segment OM and let l be the Simson line for point M, which line is constructed with respect to $\triangle ABC$. Find the locus of points P symmetric to Q with respect to l.

Answer. The segment $A'B'$ obtained by a parallel translation of diameter AB, the translation being defined by the directed line segment $\frac{1}{2}\overrightarrow{OC}$ (Fig. 50).

22. Given in the plane two distinct points $A(z_1)$ and $B(z_2)$. Find the affix p' of point P' which is symmetric to point $P(p)$ about AB.

Hint. The affix p' is found from the equation (Fig. 51)

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ p_1 & \bar{p}_1 & 1 \end{vmatrix} = 0, \text{ where } p_1 = (p + p')/2.$$

Answer. $p' = (z_1\bar{z}_2 - z_2\bar{z}_1 + \bar{p}(z_2 - z_1))/(\bar{z}_2 - \bar{z}_1)$.

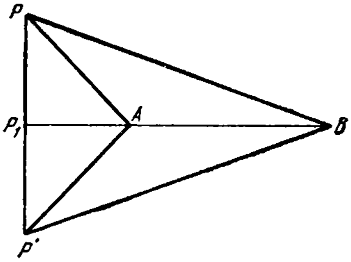

Fig. 51

23. Two distinct points $A(z_1)$ and $B(z_2)$ are given in a plane. Prove that if the point $M(\lambda)$ describes the unit circle (with the exception of a point with affix $\lambda = -1$), then the point $M'\left(\frac{z_1 + \lambda z_2}{1 + \lambda}\right)$ describes the mid-perpendicular of segment AB.

24. Let D, E, F be the points of tangency of the circle (I) inscribed in $\triangle ABC$ with sides BC, CA, AB. Denote by A_1, B_1, C_1 the midpoints of the medians DA_0, EB_0, FC_0 of the triangles ADI, BEI, CFI emanating from the vertices D, E, F (A_0, B_0, C_0 are the respective midpoints of the segments AI, BI, CI). Let H be the orthocentre of $\triangle DEF$. Prove that (Fig. 52)

$$IA_1 = \frac{IH}{4\sin(A/2)}, \quad IB_1 = \frac{IH}{4\sin(B/2)}, \quad IC_1 = \frac{IH}{4\sin(C/2)}.$$

Hint. Take (DEF) as the unit circle; denote by z_1, z_2, z_3 the affixes of the points D, E, F. Then

$$IA_1 = \frac{|\sigma_2|}{2|z_2 + z_3|}, \quad IA = \frac{2}{|z_2 + z_3|}, \quad |\sigma_2| = |\sigma_1|$$

and so on.

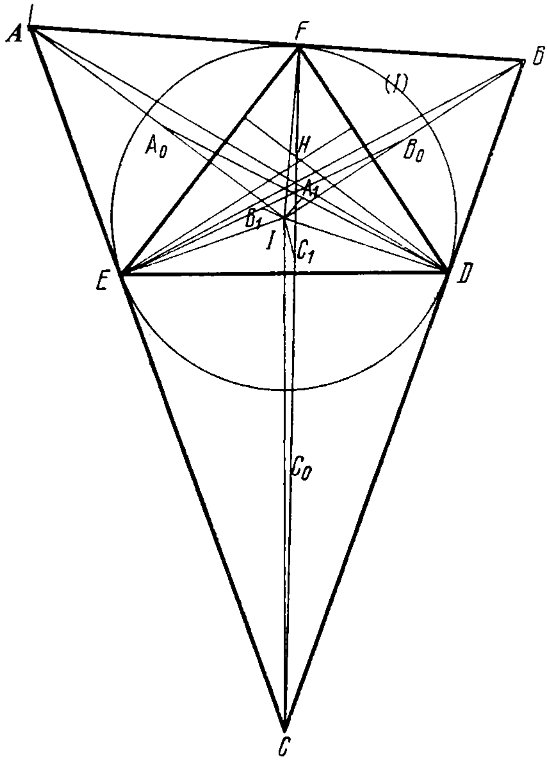

Fig. 52

25. ABC is an arbitrary triangle; $(ABC) = (O)$ is the circle circumscribed about it; Ω is an arbitrary point lying on that circle. Through point A draw a straight line parallel to line $O\Omega$ (Fig. 53). Let A' be the

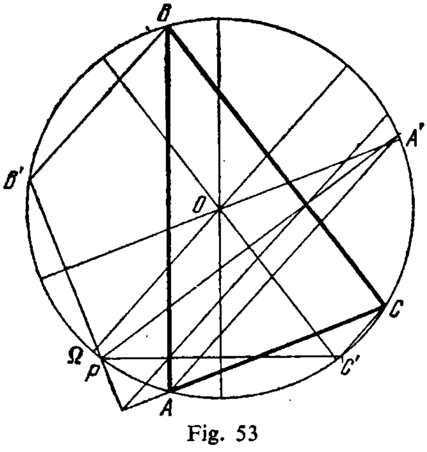

Fig. 53

second point of intersection of this line with the circle (ABC), and let P be the point symmetric to point A' about the diameter of (O) parallel to BC. Prove that:

1°. A similar construction carried out for the points B and C leads to the same point P.

2°. The Simson line of point P with respect to $\triangle ABC$ is parallel to the line OP.

26. Let $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ be four straight lines lying in the same plane; none are parallel and no three pass through one point. Any three of them, say $\Delta_1, \Delta_2, \Delta_3$, form a triangle. Suppose C_{123} is a circle circumscribed about this triangle. Construct similar circles C_{124}, C_{134}, C_{234}. Prove that:

1°. The centres of the circles C_{234}, C_{134}, C_{124}, C_{123} lie on one and the same circle C_{1234}.

2°. The circles $C_{234}, C_{134}, C_{124}, C_{123}, C_{1234}$ pass through the same point γ_4 (Fig. 54).

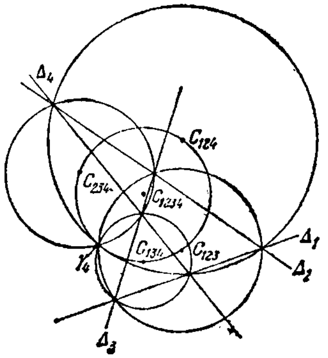

Fig. 54

To the lines $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ add a fifth straight line Δ_5, but do this so that no three lines of the five belong to a single pencil. Eliminating one of these lines, say Δ_5, we construct circle C_{1234} and, similarly, eliminating lines $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ in turn, we construct the circles C_{2345}, $C_{1345}, C_{1245}, C_{1235}$ (Fig. 55). Prove that:

3°. The centres of these five circles lie on one and the same circle C_{12345}.

4°. The circles $C_{2345}, C_{1345}, C_{1245}, C_{1235}, C_{1234}$ pass through the same point γ_5, which, however, does not generally lie on the circle C_{12345}.

Prove similar statements for any number of straight lines $\Delta_1, \Delta_2, \ldots, \Delta_n$.

Hint. Here, in brief form, is the solution to this problem given by a 9th form student of a Moscow secondary school.

Let $x_1, x_2, \ldots, x_n$ be the affixes of points symmetric to the point O (an arbitrary point taken as the origin) with respect to the straight lines $\Delta_1, \Delta_2, \ldots, \Delta_n$. Set $t_j = -\bar{x}_j/x_j$. Consider the expression

$$a_{n,i} = \frac{x_1 t_1^i}{(t_1 - t_2)(t_1 - t_3)\ldots(t_1 - t_n)} + \frac{x_2 t_2^i}{(t_2 - t_1)(t_2 - t_3)\ldots(t_2 - t_n)} + \ldots$$

From this relation it follows that

$$a_{n,i} + a_{n+1,\, i+1} - t_{n+1}\, a_{n+1,i} \tag{175}$$

and

$$\bar{a}_{n,i} = (-1)^n \sigma_n\, a_{n,n-i-1}, \tag{176}$$

where $\sigma_n = t_1 t_2 \ldots t_n$. The equation of the straight line Δ_j is

$$\bar{x} = t_j(x - x_j).$$

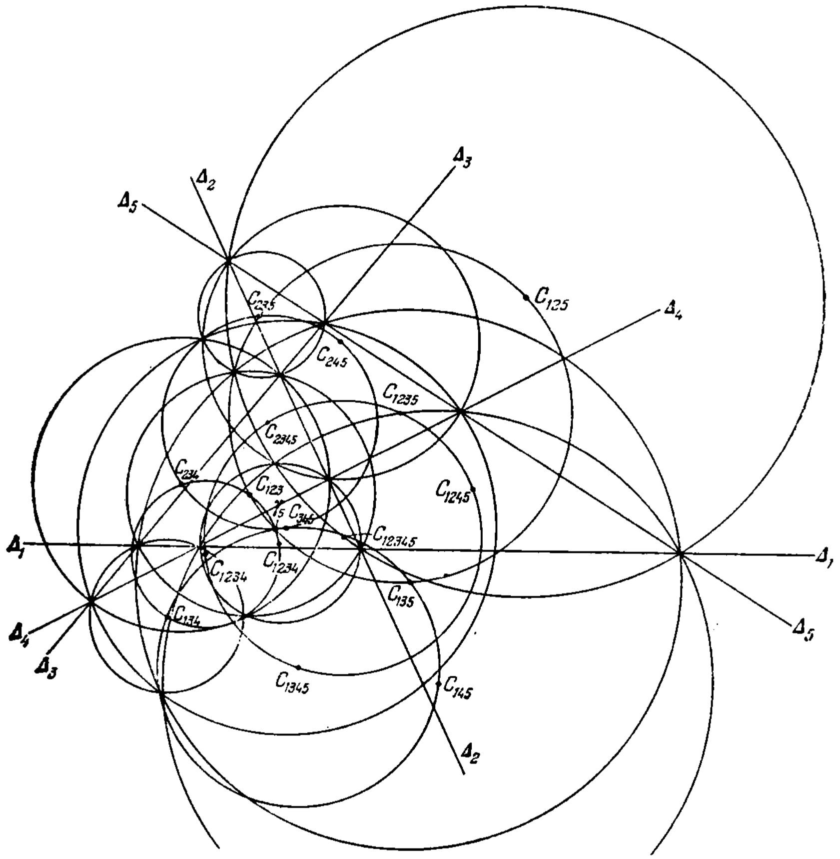

Fig. 55

The affix z_2 of the point of intersection of the straight **lines** Δ_1 and Δ_2 is

$$z_2 = \frac{t_1 x_1}{t_1 - t_2} + \frac{t_2 x_2}{t_2 - t_1} = a_{2,1} = a_{3,2} - t_3 a_{3,1}$$

and, similarly, the affixes of the points of intersection of the lines Δ_1, Δ_3 and Δ_2, Δ_3 are $a_{3,2} - t_2 a_{3,1}$ and $a_{3,2} - t_1 a_{3,1}$. From this it follows that the points of intersection of the lines Δ_1, Δ_2, Δ_3 taken in pairs lie on the circle $x = a_{3,2} - t a_{3,1}$ ($|t| = 1$), the affix of whose centre is $a_{3,2}$ and the radius is $|a_{3,1}|$. This circle is said to be a circle associated with three lines Δ_1, Δ_2, Δ_3, and its centre is a point associated with these three straight lines. Also consider the straight line Δ_4. From the relation $a_{3,2} = a_{4,3} - t_4 a_{4,2}$ it follows that the affix $a_{3,2}$ of the centre of the circle associated with the lines Δ_1, Δ_2, Δ_3 may be taken in the form $a_{4,3} - t_4 a_{4,2}$ and, similarly, the affixes

of the centres of the three circles associated with triplets of straight lines $(\Delta_1, \Delta_2, \Delta_4)$, $(\Delta_1, \Delta_3, \Delta_4,)$ $(\Delta_2, \Delta_3, \Delta_4)$ will be $a_{4,3} - t_3 a_{4,2}$, $a_{4,3} - t_2 a_{4,2}$, $a_{4,3} - t_1 a_{4,2}$. From this it follows that the centres of these four circles (circumscribed about the triangles formed by the straight lines $(\Delta_1, \Delta_2, \Delta_3)$, $(\Delta_1, \Delta_2, \Delta_4)$, $(\Delta_1, \Delta_3, \Delta_4)$, $(\Delta_2, \Delta_3, \Delta_4)$) lie on the circle $x = a_{4,3} - ta_{4,2}$, the affix of whose centre is $a_{4,3}$ and the radius is equal to $|a_{4,2}|$. We will call this circle the circle associated with four straight lines $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ and its centre the point associated with these lines.

Continuing this reasoning, we arrive at the following theorem: if n straight lines $\Delta_1, \Delta_2, \ldots, \Delta_n$ are given in a plane, then, by eliminating one line at a time from this set $\Delta_n, \ldots, \Delta_2, \Delta_1$, we obtain n groups of lines with $n-1$ lines in each group; the centres of the n circles associated with these n groups belong to one circle Γ_n:

$$x = a_{n,\,n-1} - ta_{n,\,n-2} \quad (|t| = 1), \tag{177}$$

the affix of whose centre is $a_{n,\,n-1}$ and the radius is $|a_{n,\,n-2}|$. We call this circle Γ_n the circle associated with n straight lines $\Delta_1, \Delta_2, \ldots, \Delta_n$, and its centre is called the point associated with these lines. The equation of the circle Γ_{n-1} associated with the $n-1$ lines $\Delta_1, \Delta_2, \ldots, \Delta_{n-1}$ is of the form

$$x = a_{n-1,\,n-2} - ta_{n-1,\,n-3} \quad (|t| = 1). \tag{178}$$

The complex number

$$-\left(t_n - \frac{a_{n,2}}{a_{n,1}}\right) \Big/ \left(1 - t_n \frac{a_{n,\,n-3}}{a_{n,\,n-2}}\right)$$

has a modulus equal to 1 [this follows from (176)]. If this number is substituted into (178) instead of t, we obtain

$$x = \gamma_n = a_{n,\,n-1} - \frac{a_{n,2}}{a_{n,1}} a_{n,\,n-2},$$

which is an expression symmetric in the indices $1, 2, \ldots, n$ (this is because the first position is always occupied by n). From this it follows that the n circles associated with n systems of lines having $n-1$ lines in each system (these systems are obtained by a successive elimination of one of the n lines from the system $\Delta_1, \Delta_2, \ldots, \Delta_n$) pass through a single point with affix γ_n.

Remark. For $n = 4$, the modulus of the complex number

$$\frac{a_{n,2}}{a_{n,1}} = \frac{a_{4,2}}{a_{4,1}}$$

is equal to 1, hence the circle Γ_4,

$$x = a_{4,3} - ta_{4,2},$$

associated with the four lines $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ [that is, containing the centres of the four circles circumscribed about the triangles $(\Delta_1, \Delta_2, \Delta_3)$, $(\Delta_1, \Delta_2, \Delta_4)$,

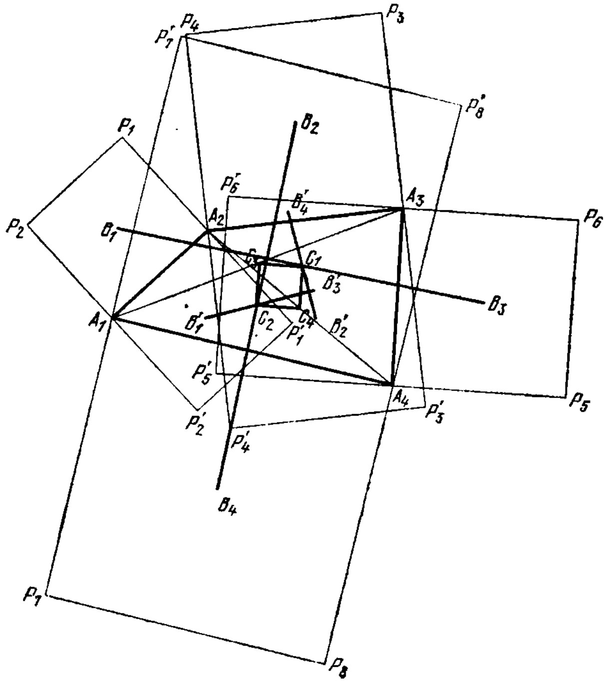

Fig. 56

$(\Delta_1, \Delta_3, \Delta_4)$, $(\Delta_2, \Delta_3, \Delta_4)$] also passes through the point with affix

$$\gamma_4 = a_{4,3} - \frac{a_{4,2}}{a_{4,1}} a_{4,2}.$$

A similar situation does not hold when $n > 4$, because, for example, the modulus of the complex number $\dfrac{a_{5,2}}{a_{5,1}}$ is no longer (generally speaking), equal to 1.

27. $A_1A_2A_3A_4$ is an arbitrary quadrangle (not necessarily convex). Squares $\overrightarrow{A_1A_2P_1P_2}$, $\overrightarrow{A_2A_3P_3P_4}$, $\overrightarrow{A_3A_4P_5P_6}$, $\overrightarrow{A_4A_1P_7P_8}$ are constructed on its sides; the squares have the same orientation; also constructed on its sides are the four squares $\overrightarrow{A_1A_2P_1'P_2'}$, $\overrightarrow{A_2A_3P_3'P_4'}$, $\overrightarrow{A_3A_4P_5'P_6'}$, $\overrightarrow{A_4A_1P_7'P_8'}$, which also have the same orientation, but the orientation of any one of the squares of the first group is opposite that of any one of the squares of the second group. Let B_1, B_2, B_3, B_4 be the centres of the squares of the first group, and let B_1', B_2', B_3', B_4' be the centres of the squares of the second group. Prove that:

1°. Segments B_1B_3 and B_2B_4 are equal and mutually perpendicular (the quadrangle $B_1B_2B_3B_4$ that satisfies this condition is termed a *pseudo-square*).

2°. $B_1'B_2'B_3'B_4'$ is also a pseudosquare.

3°. The segments B_1B_3 and $B_2'B_4'$ have the same midpoint C_1; the segments $B_1'B_3'$ and B_2B_4 have the same midpoint C_2. Let C_3 and C_4 be the respective midpoints of the segments A_1A_3 and A_2A_4. Prove that $C_1C_2C_3C_4$ is a square (Fig. 56).

[Solve this problem by using complex numbers; also solve it by the methods of analytic geometry and the methods of vector algebra.]

28. $A_1A_2A_3A_4A_5A_6A_7A_8$ is an arbitrary octagon (not necessarily convex). On its sides, construct the squares

$$\overrightarrow{A_1A_2P_1P_2},\ \overrightarrow{A_2A_3P_3P_4},\ \overrightarrow{A_3A_4P_5P_6},\ \overrightarrow{A_4A_5P_7P_8},$$

$$\overrightarrow{A_5A_6P_9P_{10}},\ \overrightarrow{A_6A_7P_{11}P_{12}},\ \overrightarrow{A_7A_8P_{13}P_{14}},\ \overrightarrow{A_8A_1P_{15}P_{16}}$$

so that all of them have the same orientation, and also construct another eight squares

$$\overrightarrow{A_1A_2P_1'P_2'},\ \overrightarrow{A_2A_3P_3'P_4'},\ \overrightarrow{A_3A_4P_5'P_6'},\ \overrightarrow{A_4A_5P_7'P_8'},$$

$$\overrightarrow{A_5A_6P_9'P_{10}'},\ \overrightarrow{A_6A_7P_{11}'P_{12}'},\ \overrightarrow{A_7A_8P_{13}'P_{14}'},\ \overrightarrow{A_8A_1P_{15}'P_{16}'},$$

also having the same orientation and such that the orientation of any one of the squares of the first group is opposite that of any one of the squares of the second group. Let $B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8$ be the respective centres of the squares of the first group, and $B_1', B_2', B_3', B_4', B_5', B_6', B_7', B_8'$ the respective centres of the squares of the second group.

Prove that:

1°. The midpoints C_1, C_2, C_3, C_4 of the principal diagonals * of the octagon $B_1B_2B_3B_4B_5B_6B_7B_8$ form a pseudosquare.

2°. The midpoints C_1', C_2', C_3', C_4' of the principal diagonals of the octagon $B_1'B_2'B_3'B_4'B_5'B_6'B_7'B_8'$ also form a pseudosquare.

3°. The sets of points

$$A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8,$$
$$B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8,$$
$$C_1, C_2, C_3, C_4,$$
$$C_1', C_2', C_3', C_4'$$

have a common centroid.

4°. The midpoints of the segments C_1C_3 and C_2C_4 coincide respectively with the midpoints of the segments $C_2'C_4'$ and $C_1'C_3'$.

* The principal diagonals of the octagon $B_1B_2B_3B_4B_5B_6B_7B_8$ are the segments B_1B_5, B_2B_6, B_3B_7, B_4B_8 that join opposite vertices (and also the straight lines on which the segments lie).

5°. A necessary and sufficient condition that $C_1C_2C_3C_4$ and, hence, also $C_1'C_2'C_3'C_4'$ be squares (with the indicated order of their vertices) is that the two sets of points A_1, A_3, A_5, A_7 and A_2, A_4, A_6, A_8 have a common centroid.

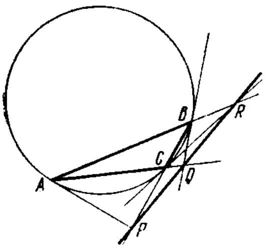

Fig. 57

6°. The squares $C_1C_2C_3C_4$ and $C_1'C_2'C_3'C_4'$ coincide ($C_1 = C_1'$, $C_2 = C_2'$, $C_3 = C_3'$, $C_4 = C_4'$) if and only if the two sequences of the principal diagonals of the octagon $A_1A_2A_3A_4A_5A_6A_7A_8$ are bisected by their points of intersection.

7°. If the points C_1, C_3, C_2', C_4' and the points C_1', C_3', C_2, C_4 coincide, then the quadrangles $A_1A_3A_5A_7$ and $A_2A_4A_6A_8$ are parallelograms, and conversely.

Solve this problem through the use of (1) complex numbers, (2) vector algebra, and (3) analytic geometry.

29. Inscribed in a circle (O) is a triangle ABC. Prove that the points P, Q, R of intersection of the tangent lines, drawn at the points A, B, C to (O), with the straight lines BC, CA, AB, respectively, lie on one straight line (Fig. 57).

30. Through points $A = (z_1)$, $B = (z_2)$, $C = (z_3)$ lying on the unit circle draw straight lines parallel to a given straight line Δ, which has slope λ. Let A_1, B_1, C_1 be the second points of intersection of these lines with the circle (ABC); A_2, B_2, C_2 are points symmetric to the points A_1, B_1, C_1 with respect to the straight lines BC, CA, AB respectively. Draw through points A_2, B_2, C_2 straight lines parallel to the line Δ, and denote the points of intersection of these lines with the lines BC, CA, AB respectively by P, Q, R. Prove that the points P, Q, R lie on one straight line, and set up the equation of that line (Fig. 58).

Answer. $\lambda(2\lambda^3 + \sigma_1 \lambda^2 - 1) z + \lambda^2(2 + \sigma_1 \lambda - \lambda^3) \bar{z}$

$$= \lambda^6 + \sigma_2 \lambda^5 - \sigma_1 \lambda^4 + (1 - \sigma_1 \sigma_2) \lambda^3 - \sigma_2 \lambda^2 + \sigma_1 \lambda + 1.$$

31. Four points A_1, A_2, A_3, A_4 are taken on a circle (O). Let P be the point of intersection of the lines A_1A_3 and A_2A_4; let Q be the point of intersection of the lines A_1A_2 and A_3A_4; let R be the point of intersection of the tangents to (O) at the points A_1 and A_4; let S be the point of intersection of the tangents to the circle (O) at the points A_2 and A_3. Prove that the points P, Q, R, S lie on one straight line (Fig. 59).

Hint. Take (O) as the unit circle.

32. A triangle ABC is inscribed in a circle (O). Tangent lines are drawn to (O) at the points A, B, C; these lines form $\triangle A_0B_0C_0$ (called the *tangential triangle* of $\triangle ABC$). Prove that:

1°. The points P, Q, R of intersection of the straight lines: $P = (BC, B_0C_0)$ $Q = (CA, C_0A_0)$, $R = (AB, A_0B_0)$ lie on one straight line m.

2°. The lines AA_0, BB_0, CC_0 pass through the single point M.

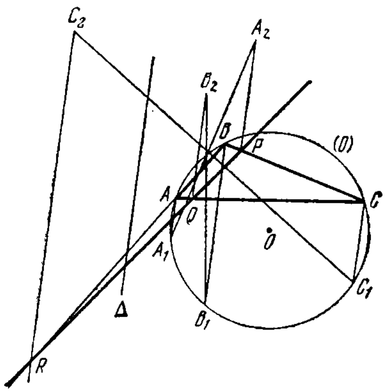

Fig. 58

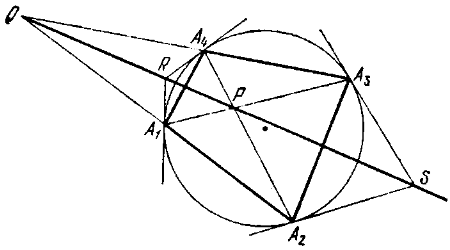

Fig. 59

3°. The point M is the pole of the straight line m with regard to the circle (O) (or, what is the same, m is the polar of point M with respect to this circle) (Fig. 60).

Hint. Take (O) for the unit circle.

33. $A_1A_2A_3$ is an arbitrary triangle lying in an oriented plane; P is an arbitrary point lying in the plane of that triangle but not lying on any one of its sides, nor on the circle $(A_1A_2A_3)$. Let B_1, B_2, B_3 be the orthogonal projections of point P on the lines A_2A_3, A_3A_1, A_1A_2; C_1, C_2, C_3 are the orthogonal projections of point P on the straight lines B_2B_3, B_3B_1, B_1B_2, and D_1, D_2, D_3 are the orthogonal projections of point P on the straight lines C_2C_3, C_3C_1, C_1C_2 respectively. Prove that $\triangle \overrightarrow{A_1A_2A_3}$ and $\triangle \overrightarrow{D_1D_2D_3}$ are similar and have the same orientation (Fig. 61).

Hint. Take point P for the coordinate origin. Let a_k, b_k, c_k, d_k ($k = 1, 2, 3$) be the respective affixes of the points A_k, B_k, C_k, D_k. Express b_k in terms of a_k; c_k in terms of b_k; d_k in terms of c_k.

34. On the circle (O) take six arbitrary points A_1, A_2, A_3, A_4, A_5, A_6. Prove that the three points P, Q, R,

$$P = (A_1A_2, A_4A_5), \quad Q = (A_2A_3, A_5A_6), \quad R = (A_3A_4, A_6A_1),$$

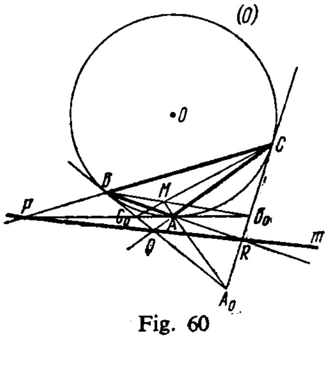

Fig. 60

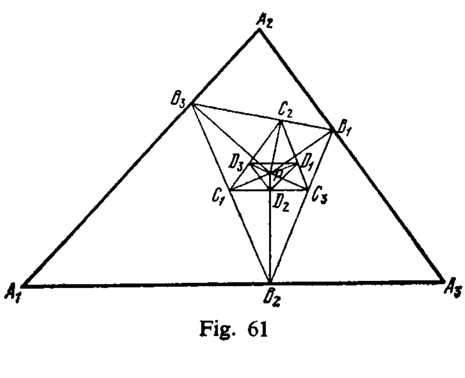

Fig. 61

the points of intersection of the straight lines, lie on one straight line *(Pascal's theorem)* (Fig. 62).

Hint. Take (O) for the unit circle.

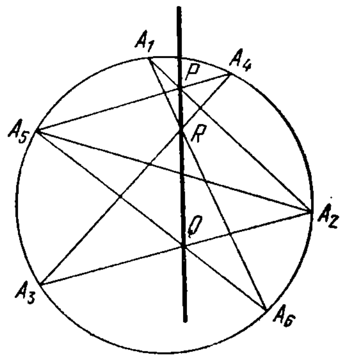

Fig. 62

35. Prove that the points $A(z_1)$ and $B(z_2)$, where z_1 and z_2 are the roots of the equation

$$z^2 + 2pz + q = 0$$

(p and q are complex numbers and $p^2 - q \neq 0$), lie on the straight line passing through the origin of coordinates if and only if one of the following conditions is valid:

(1) $p = 0$;

(2) $p \neq 0$, q/p^2 is a real number and $(q/p^2) < 1$; here, if $p \neq 0$, $0 \leqslant q/p^2 < 1$, then the points A and B lie on one ray emanating from the coordinate origin O; and if $p \neq 0$, $q/p^2 < 0$, then the points A and B lie on opposite rays emanating from the origin.

36. Given a cubic equation:

$$a_0 z^3 + 3a_1 z^2 + 3a_2 z + a_3 = 0.$$

Let z_1, z_2, z_3 be its roots. Prove that a necessary and sufficient condition that the points $A(z_1), B(z_2), C(z_3)$ be collinear is as follows: either

(1) $2a_1^3 - 3a_0 a_1 a_2 + a_0^2 a_3 = 0$

or

(2) $\dfrac{(a_0 a_2 - a_1^2)^3}{(2a_1^3 - 3a_0 a_1 a_2 + a_0^2 a_3)^2}$ which is a real number less than $-1/4$.

Hint. Under the translation transformation

$$z = \zeta - \frac{a_1}{a_0}$$

(Tschirnhaus transformation), the given equation takes the form

$$\zeta^3 + 3p\zeta + q = 0,$$

where

$$p = \frac{a_0 a_2 - a_1^2}{a_0^2}, \quad q = \frac{2a_1^3 - 3a_0 a_1 a_2 + a_0^2 a_3}{a_0^3}.$$

The sum of the roots $\zeta_1, \zeta_2, \zeta_3$ of the last equation is zero, and since $(\zeta_1 + \zeta_2 + \zeta_3)/3 = 0$ is the affix of the centroid of the system of points $A'(\zeta_1), B'(\zeta_2), C'(\zeta_3)$, it follows that this centroid coincides with the coordinate origin. If the points A, B, C are collinear, then the points A', B', C'

are also collinear (and conversely). If the points A', B', C' are collinear, then, by the foregoing, they lie on a straight line that passes through the origin.

37. Let a general Cartesian system of coordinates Oxy be introduced in the plane. The *affine transformation of the plane* is a correspondence under which the coordinates x', y' of the image $M'(x', y')$ of a point $M(x, y)$ are expressed in terms of the coordinates x, y of the preimage $M(x, y)$ of the point $M'(x', y')$ by the linear relations

$$x' = a_1x + b_1y + c_1,$$
$$y' = a_2x + b_2y + c_2,$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are arbitrary real numbers and $a_1b_2 - a_2b_1 \neq 0$.

Prove that:

1°. Every affine transformation may be written as

$$z' = \alpha z + \beta\bar{z} + \gamma,$$

where z and z' are the respective affixes of the points M and M', and $|\alpha| \neq |\beta|$. Conversely, if $|\alpha| \neq |\beta|$, then the relation $z' = \alpha z + \beta\bar{z} + \gamma$ defines a certain affine transformation.

2°. There exists an affine transformation (and only one) which carries any three noncollinear points $A(z_1)$, $B(z_2)$, $C(z_3)$ into the three noncollinear points $A'(z_1')$, $B'(z_2')$, $C'(z_3')$, respectively.

3°. A triangle $\overrightarrow{ABC}$ is said to be *metaparallel* to $\triangle\overrightarrow{A'B'C'}$ if the straight lines that pass through the points A, B, C and are parallel respectively to the lines $B'C', C'A', A'B'$ intersect in one point. Let $z' = \alpha z + \beta\bar{z} + \gamma$ be an affine transformation under which $\triangle\overrightarrow{ABC}$ goes into $\triangle\overrightarrow{A'B'C'}$. Prove that $\triangle\overrightarrow{ABC}$ is metaparallel to $\triangle\overrightarrow{A'B'C'}$ if and only if:

(1) α is a pure imaginary number or, what is the same,

(2) $$\Delta = \begin{vmatrix} z_1 & \bar{z}_1' & 1 \\ z_2 & \bar{z}_2' & 1 \\ z_3 & \bar{z}_3' & 1 \end{vmatrix}$$

is a real number.

4°. Prove that the concept of metaparallelism is symmetric, but not reflexive and not transitive.

5°. Prove that if $\triangle\overrightarrow{ABC}$ is metaparallel to two of the three triangles $\overrightarrow{A'B'C'}$, $\overrightarrow{B'C'A'}$, $\overrightarrow{C'A'B'}$, then it is metaparallel to the third one as well (in this case we say that $\triangle\overrightarrow{ABC}$ and $\triangle\overrightarrow{A'B'C'}$ are three-times metaparallel).

6°. Given a triangle ABC in a plane:

$$A = (z_1), \quad B = (z_2), \quad C = (z_3).$$

Find the affix z_3' of point C' if we know that $\triangle\overrightarrow{ABC}$ is three-times meta-parallel to $\triangle\overrightarrow{A'B'C'}$ with vertices

$$A' = (0), \quad B' = (1), \quad C' = (z_3').$$

7°. A triangle $\overrightarrow{ABC}$ is said to be *orthologic* to $\triangle\overrightarrow{A'B'C'}$ if the straight lines that pass through the points A, B, C and are respectively perpendicular to the straight lines $B'C', C'A', A'B'$ intersect in one point. Prove that $\triangle\overrightarrow{ABC}$ is orthologic to $\triangle\overrightarrow{A'B'C'}$ if and only if

(1) α is a real number, or, what is the same,

(2) $$\begin{vmatrix} z_1 & \bar{z}_1' & 1 \\ z_2 & \bar{z}_2' & 1 \\ z_3 & \bar{z}_3' & 1 \end{vmatrix}$$

is a pure imaginary number.

8°. Prove that the concept of orthologicality is reflexive and symmetric but is not transitive.

9°. Prove that if $\triangle\overrightarrow{ABC}$ is orthologic to two of the triangles $\overrightarrow{A'B'C'}$, $\overrightarrow{B'C'A'}$, $\overrightarrow{C'A'B'}$, then it is orthologic also to the third triangle (in this case we say that $\triangle\overrightarrow{ABC}$ and $\triangle\overrightarrow{A'B'C'}$ are three-times orthologic).

Answer. 6°. $$z_3' = \frac{\begin{vmatrix} \bar{z}_1 - z_1 & z_2 & 1 \\ \bar{z}_2 - z_2 & z_3 & 1 \\ \bar{z}_3 - z_3 & z_1 & 1 \end{vmatrix}}{\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}}.$$

Chapter IV

INVERSION

Sec. 1. Inversion defined. Properties of inversion

Let us adjoin to the set of all points of the Euclidean plane a single element which we will call the *ideal point* or the *point at infinity* of the plane π. Let us agree that any straight line of the π-plane passes through the point at infinity and that this point does not belong to any finite figure. The Euclidean plane supplemented by a single point at infinity (with the indicated agreements) is termed a *Euclidean circular plane* or, simply, a *circular plane*. Straight lines lying in the circular plane will sometimes be called circles of infinite radius. We will also regard points as circles; such "circles" will be called circles of zero radius or zero circles. Two intersecting straight lines have two common points, one proper point and the other the point at infinity. Two parallel lines have only one point in common: the point at infinity; we will say that two parallel straight lines or two circles of infinite radius meet at the point at infinity.

Suppose O is a fixed proper point of a circular plane, and k is a fixed real number not zero. An *inversion* $[O, k]$ *with pole* O and *power* k of the π-plane is a one-to-one transformation of that plane under which each proper point M of the π-plane distinct from point O is associated with a proper point M' lying on the straight line OM and such that

$$\overrightarrow{OM}\cdot\overrightarrow{OM'} = (OM)(OM') = k.$$

With the pole O of inversion we associate the point at infinity O', and with the point O' we associate the point O.

Every inversion I is an *involutory transformation*, that is,

$$I^2 = E,$$

where E is the identical transformation.

This follows from the fact that if under inversion I a point M' is the image of a point M, then M is the image of M'.

If the power k of the inversion $[O, k]$ is positive, then the circle K with centre O and radius $\sqrt{k}$ is termed the *circle of inversion*, and the inversion itself is also called a symmetry with respect to the circle K. Under the inversion $[O, k]$, where $k > 0$, each point M of the circle of inversion is invariant, that is, its image M' coincides with the point itself.

To construct the image M' of point M lying outside the circle K of inversion $[O, k]$, where $k > 0$, we draw from point M a tangent MT to the circle K (T is the point of tangency); M' is the projection of T on the straight line OM (two tangents may be drawn, MT and MT', the

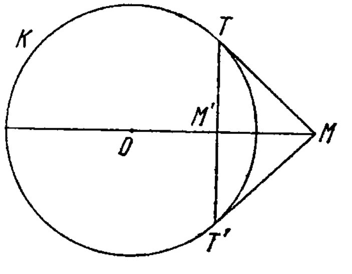

Fig. 63

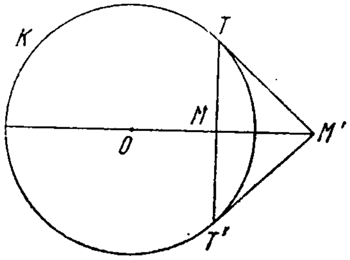

Fig. 64

point M' is the point of intersection of straight lines OM and TT') (Fig. 63).

To construct the image M' of point M lying inside the circle K of inversion $[O, k]$, $k > 0$, draw through M a straight line perpendicular to the straight line OM. Let T be any one of the points of intersection of this perpendicular with the circle K. Then the tangent to the circle K at the point T will intersect the straight line OM in the point M' (Fig. 64).

Thus, under the positive inversion $[O, k]$, $k > 0$, points lying outside the circle of inversion K pass into interior points of that circle, and points lying inside the circle K pass into points lying outside the circle.

For negative inversion $[O, k]$, $k < 0$, the circle K with centre O and radius $\sqrt{|k|}$ is invariant; however, each one of its points is noninvarian and passes into a point diametrically opposite it. To construct the image M of a point M under negative inversion $[O, k]$, $k < 0$, we construct the image M^* of point M under the negative inversion $[O, |k|]$, M' is symmetric to point M^* with respect to point O. From this it follows that under negative inversion as well the set of all points lying outside the circle K passes into the set of interior points of circle K and the set of all interior points of circle K goes into the set of all exterior points of K (Fig. 65 and Fig. 66).

Let $[O, k]$ be an inversion with pole O and power k. If C is an arbitrary circle that does not pass through the pole of inversion, then its image C' under the inversion $[O, k]$ is again a circle that does not pass through the pole of inversion.

Proof. Let M be an arbitrary point of the circle C, and let M' be its image under the inversion $[O, k]$, that is, $\overrightarrow{OM} \cdot \overrightarrow{OM'} = k$; here it is well to point out that there is a frequently used notation $\overline{OM} \cdot \overline{OM'} = k$, where we speak of the product of the lengths (OM) and (OM') of directed line segments. Since the straight line OM has a common point M with the circle C, this line has a second common point N (possibly $M = N$) with the circle C. The product $\overrightarrow{OM} \cdot \overrightarrow{ON} = \overline{OM} \cdot \overline{ON} = \sigma$ is the power of the pole O of inversion with respect to the circle C (Fig. 67). From

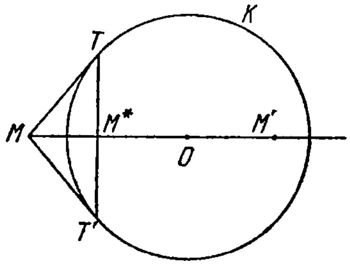

Fig. 65

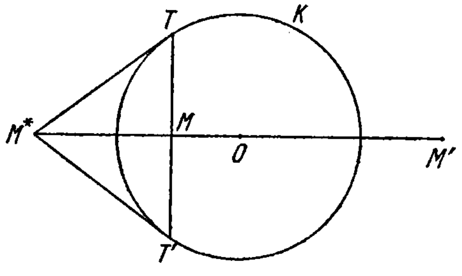

Fig. 66

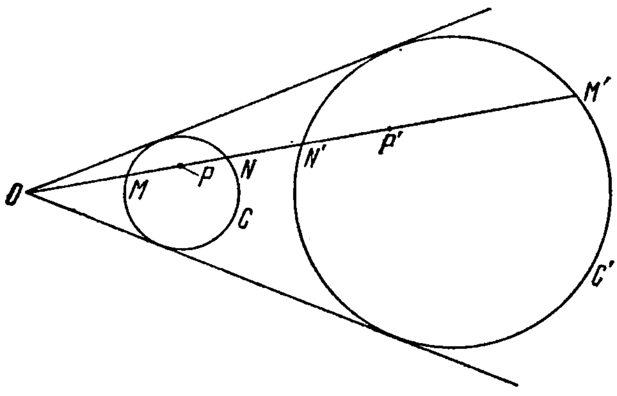

Fig. 67

the last two equations it follows that

$$\frac{\overrightarrow{OM'}}{\overrightarrow{ON}} = \frac{k}{\sigma}, \qquad \overrightarrow{OM'} = \frac{k}{\sigma}\,\overrightarrow{ON}.$$

Thus, the point M' is the image of point N of the circle C under the homothetic transformation $(O, k/\sigma)$. But under the homothetic transformation $(O, k/\sigma)$, circle C goes into circle C'. If point M describes a circle C, then the point N too describes the same circle C, and, hence, its image M' under the homothetic transformation $(O, k/\sigma)$ describes the circle C'. Thus, the circle C' is the image of C also under the homothetic transformation $(O, k/\sigma)$, where σ is the power of the pole O of inversion with respect to the circle C^* and under the inversion $[O, k]$.

* Note that the inversion $[O, k]$ and the homothetic transformation $(O, k/\sigma)$ carry circle C into one and the same circle C'; however, the points lying on circle C are transformed by the inversion $[O, k]$ and the homothetic transformation $(O, k/\sigma)$ into different points of circle C': under the inversion $[O, k]$, point M goes into point M' of circle C', and under the homothetic transformation $(O, k/\sigma)$, point M goes into point N', where N' is the second point of intersection of straight line OM' and circle C'.

If circle C passes through the pole of inversion O, then its image under the inversion $[O, k]$ is a straight line that does not pass through the pole of inversion and is perpendicular to the straight line joining the pole of inversion and the centre of circle C.

If the straight line does not pass through the pole of inversion O, then its image under the inversion $[O, k]$ is a circle that passes through the pole of inversion.

If the straight line passes through the pole of inversion O, then under the inversion $[O, k]$ it passes into itself.

Under inversion, the tangency of the circles is retained.

This follows from the fact that under inversion, a circle passes into a circle and the inversion is a one-to-one transformation (to explain this more fully: under inversion, two tangent circles can go into two tangent circles or into a circle and a tangent line to the circle, or into two parallel straight lines).

If the pole of inversion $[O, k]$ lies outside circle C, and C' is the inverse, then the set of points lying inside circle C goes into the set of points lying inside circle C' (and conversely). The set of points lying outside (C) goes into the set of points lying outside (C') (and conversely) (see Fig. 67).

If the pole O of inversion $[O, k]$ lies inside circle C, and C' is the inverse of (C), then the set of points lying inside (C) goes into the set of points lying outside circle C', and the set of points lying outside (C) goes into the set of points lying inside circle C'.

Remark. When investigating the mapping of regions under an inversion, it is useful to bear in mind the following. Let us consider the inversion $[O, k]$ where $k > 0$. Let M be an arbitrary proper point of the circular π-plane, and let M' be its image under the inversion $[O, k]$, that is,

$$(\overrightarrow{OM}\cdot\overrightarrow{OM'}) = OM\cdot OM' = k.$$

From this relation it follows that if the point M moves along the ray OM receding from the pole, then the point M' will move in the opposite direction since the product $\overline{OM}\cdot\overline{OM'}$ must remain constant and equal to k (the points M and M' will meet on the circle of inversion). The same occurs in the case of the opposite ray to the one we considered. In concrete cases (see below), this reasoning may be utilized profitably when investigating the mapping of regions under an inversion and when finding invariant regions, that is, regions that go into themselves under the inversion in question.

The angle between two intersecting circles is preserved under an inversion, but the orientation of the angle is reversed (inversion is a conformal transformation of the second kind). The proof of this proposition is given below when we consider the inversion of space.

Remark. A linear fractional transformation of the plane of a complex variable

$$z' = \frac{az + b}{cz + d}, \quad ad - bc \neq 0, \quad c = 0$$

may be rewritten in the form

$$z' = \frac{a}{c} + \frac{\Delta}{z + \frac{d}{c}}, \text{ where } \Delta = \frac{bc - ad}{c^2}.$$

It may be interpreted geometrically as follows:

1°. $z_1 = z + \frac{d}{c}$ is a translation.

2°. $z_2 = \frac{1}{z_1}$ is an inversion with respect to the unit circle with subsequent symmetry with respect to the x-axis *.

3°. $z_3 = \Delta z_2$ is a similarity transformation with centre O: a rotation about the origin of coordinates through an angle $\arg \Delta$ and a homothetic transformation $(O, |\Delta|)$.

4°. $z' = z_3 + \frac{a}{c}$ is a translation again.

From this it follows that a linear fractional transformation is a conformal transformation of the first kind (that preserves orientation of angles), since the transformations 1°, 3°, 4° preserve orientation of angles, and the inversion $z_2 = \frac{1}{z_1}$ with subsequent symmetry with respect to the x-axis also preserves orientation of angles.

Sec. 2. Problems involving inversion

Problem 1. Let A' and B' be images of the points A and B under the inversion $[0, k]$. Express the length of segment $A'B'$ in terms of the lengths of the segments AB, OA, OB and in terms of k. It is assumed that the points A and B are distinct from point O.

Solution. Suppose that the points O, A and B do not lie on one straight line. Let $k > 0$. Then the points A' and B' lie, respectively, on the rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$, and we have

$$\overline{OA} \cdot \overline{OA'} = \overline{OB} \cdot \overline{OB'} = k.$$

* $|z_2|\,|z_1| = 1$, $\arg z_2 \equiv -\arg z_1 \pmod{2\pi}$.

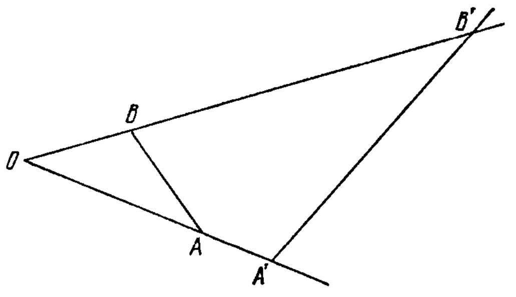

Fig. 68

From this it follows that

$$\frac{OA'}{OB} = \frac{OB'}{OA}.$$

Hence, $\triangle \overrightarrow{OAB}$ and $\triangle \overrightarrow{OB'A'}$ are similar (but with opposite orientations) (Fig. 68). From the similarity of these triangles it follows that

$$\frac{A'B'}{AB} = \frac{OA'}{OB} = \frac{OA'\cdot OA}{OB\cdot OA} = \frac{\overline{OA}\cdot\overline{OA'}}{OA\cdot OB} = \frac{k}{OA\cdot OB} = \frac{|k|}{OA\cdot OB},$$

and, hence,

$$A'B' = \frac{|k|AB}{OA\cdot OB}.$$

This formula is also true if the points O, A and B lie on one straight line and in the case of $k < 0$.

Problem 2. Prove that it is possible to circumscribe a circle about a convex quadrangle $ABCD$ if and only if the product of the diagonals of the quadrangle is equal to the sum of the products of its opposite sides *(Ptolemy's theorem)*:

$$AC\cdot BD = AB\cdot CD + BC\cdot AD.$$

I. Suppose we can circumscribe a circle K about a quadrangle $ABCD$; we will then prove that the relation $AC\cdot BD = AB\cdot CD + BC\cdot AD$ holds true.

Consider the inversion $[A, 1]$. Under this inversion, the circle K goes into the straight line K', and the points B, C, D go into the points B', C', D' lying on that straight line. The point C' will lie between points B' and D', and therefore (Fig. 69)

$$B'C' + C'D' = B'D'$$

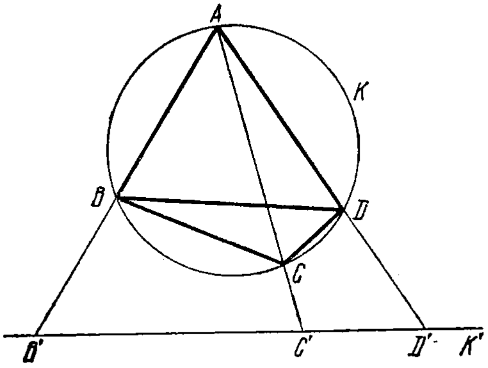

Fig. 69

or, on the basis of the preceding problem,

$$\frac{BC}{AB\cdot AC}+\frac{CD}{AC\cdot AD}=\frac{BD}{AB\cdot AD},$$

whence follows the relation $AC\cdot BD = AB\cdot CD + BC\cdot AD$.

II. Suppose the relation $AC\cdot BD = AB\cdot CD + BC\cdot AD$ is valid. Let us then prove that the quadrangle $ABCD$ is convex and a circle can be circumscribed about it.

We consider the inversion $[A, 1]$. Let B', C', D' be the inverses of the points B, C, D. From the relation

$$AC\cdot BD = AB\cdot CD + BC\cdot AD$$

we have

$$\frac{BC}{AB\cdot AC}+\frac{CD}{AC\cdot AD}=\frac{BD}{AB\cdot AD},$$

and from this relation and the result of problem 1 it follows that $B'C' + C'D' = B'D'$. Hence, the points B', C', D' lie on one straight line K', and the point C' lies between the points B' and D'. From this it follows that the points B, C, D lie on one circle (that passes through point A), which is the image of the straight line K' under the inversion $[A, 1]$. Since the ray $\overrightarrow{AC}$ lies inside $\angle A$ formed by the rays $\overrightarrow{AD}$ and $\overrightarrow{AB}$, it follows that AC is a diagonal of the quadrangle $ABCD$, and this means that the quadrangle is convex (under the indicated order of its vertices).

Problem 3. Inscribed in a circle K is an equilateral triangle ABC. Let O be a point not lying on the circle K. Prove that there is a triangle with sides OA, OB and OC. Prove that if the point O lies on the circle K, then the sum of two of its segments OA, OB, OC is equal to the third.

Proof. Suppose O does not lie on the circle K (Fig. 70). Consider the inversion $[O, 1]$. Circle K goes into circle K', and the points A, B, C go

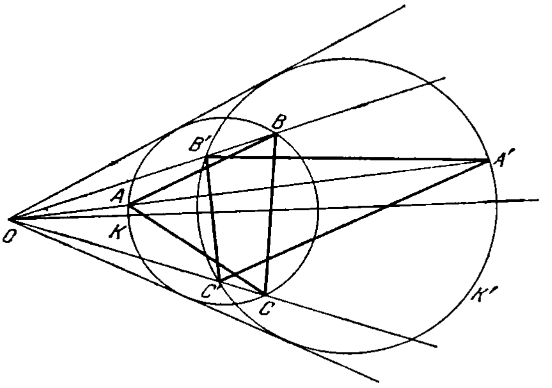

Fig. 70

into the points A', B', C' that lie on K' and, hence, do not lie on one straight line. On the basis of problem 1, we have

$$\left.\begin{aligned} B'C' &= \frac{BC}{OB\cdot OC} = \frac{OA\cdot BC}{OA\cdot OB\cdot OC}, \\ C'A' &= \frac{CA}{OC\cdot OA} = \frac{OB\cdot CA}{OA\cdot OB\cdot OC}, \\ A'B' &= \frac{AB}{OA\cdot OB} = \frac{OC\cdot AB}{OA\cdot OB\cdot OC} \end{aligned}\right\} \quad (1)$$

and, hence,

$$B'C' : C'A' : A'B' = OA : OB : OC$$

since $BC = CA = AB$. Thus, the segments OA, OB, OC are proportional to the sides $B'C'$, $C'A'$, $A'B'$ of $\triangle A'B'C'$ and, hence, there is a triangle with sides OA, OB, OC (this triangle is similar to $\triangle A'B'C'$).

Suppose point O lies on the circle K, for example, on the arc $\overset{\frown}{AC}$, that does not contain B (Fig. 71). Under the inversion $[O, 1]$, circle K goes into line K', the points A, B, C go into the points A', B', C' that lie on K', and the point B' will lie between points A' and C'; thus,

$$A'B' + B'C' = O'C'$$

and since relation (1) is valid in this case as well, it follows that

$$OB = OA + OC.$$

Remark. The theorem holds if point O is chosen arbitrarily in space. The proof is analogous.

Problem 4. Two circles C_1 and C_2 with centres O_1 and O_2 are externally tangent to each other. The straight line l touches both circles at distinct

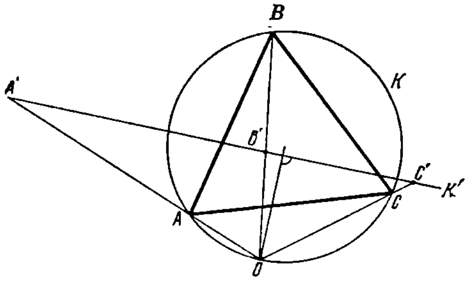

Fig. 71

points A and B. Construct a circle tangent to the two given circles and the straight line l.

Solution. Let us consider the inversion $[A, AB^2]$. Under this inversion, circle C_2 goes into itself because if an arbitrary straight line is drawn through point A, the line intersecting circle C_2 in points M and M', then $AM \cdot AM' = AB^2$. Circle C_1 goes into line C_1' parallel to line l and tangent to circle C_2 (Fig. 72). Thus, the problem reduces to constructing a circle that is tangent to circle C_2 and to two tangent lines l and C_1' parallel to it. There are two such circles. Let K_1' be one of these circles, let A_1' be the point of tangency of circles K_1' and C_2, and let B_1' be the point of tangency of K_1' to line C_1'. Let A_1 be the second point of intersection of line AA_1' with circle C_2; point A_1 is the image of point A_1' under the inversion $[A, AB^2]$. Let B_1 be the point of intersection of line AB_1' with circle C_1 (point B_1 is distinct from point A); point B_1 is the inverse of point B_1'. The points A_1 and B_1 are points of contact of the desired circle with the circles C_2 and C_1 respectively. The centre P_1 of one of the desired circles is the point of intersection of the straight lines O_2A_1 and O_1B_1, and the radius is equal to $P_1A_1 = P_1B_1$. The second circle is constructed in similar fashion.

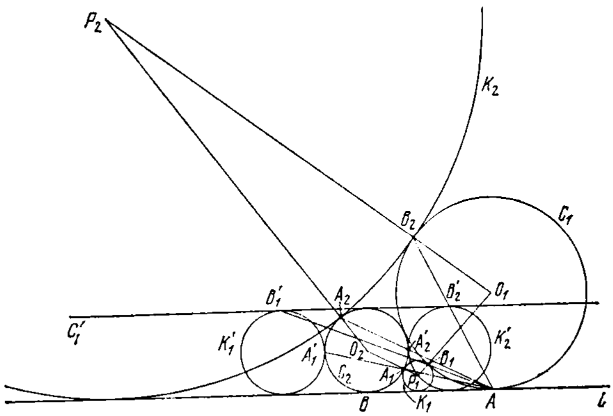

Fig. 72

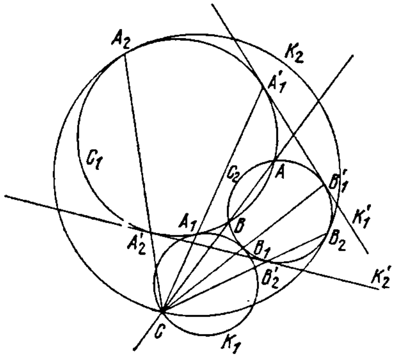

Fig. 73

Problem 5. Two circles C_1 and C_2 intersect in points A and B. A point C, different from A and B and lying outside circles C_1 and C_2, is taken on a straight line AB. Construct a circle that passes through point C and is tangent to circles C_1 and C_2 (Fig. 73).

Solution. Consider the inversion $[C, k]$, where $k = CA \cdot CB$. Under this inversion, each of the circles C_1 and C_2 goes into itself, and the desired circle goes into a straight line tangent to the circles C_1 and C_2. Since the two intersecting circles C_1 and C_2 have two common tangents K_1' and K_2', it follows the problem has two solutions. Let A_1' and B_1' be the points of tangency of line K_1' with circles C_1 and C_2. Denote by A_1 the second point of intersection of line CA_1' with circle C_1, and by B_1 the second point of intersection of line CB_1' with circle C_2. One of the desired circles passes through points C, A_1 and B_1. The second circle is constructed in similar fashion (Fig. 73).

Problem 6. Given in a $\triangle ABC$ the radius r of an inscribed circle and the radius R of a circumscribed circle. Find the distance d between their centres.

Solution. If, under the inversion $[O, k]$, circle C goes into circle C', then circle C goes into circle C' also under the homothetic transformation $(O, k/\sigma)$, where σ is the power of the point O with respect to the circle C.

Consider the inversion $[P, r^2]$, where P is the centre of the circle inscribed in the given triangle, and r is the radius. Under this inversion, the points of the inscribed circle are fixed (since the inscribed circle is the circle of inversion). The vertices of the given triangle under the inversion $[P, r^2]$ go into the midpoints of the sides of $\triangle A_1B_1C_1$, whose vertices are the points of contact of the sides of $\triangle ABC$ with the circle inscribed in $\triangle ABC$ (Fig. 74). The radius of the circle passing through the midpoints of the sides of the indicated triangle is equal to $r/2$. Thus, the circle (ABC)

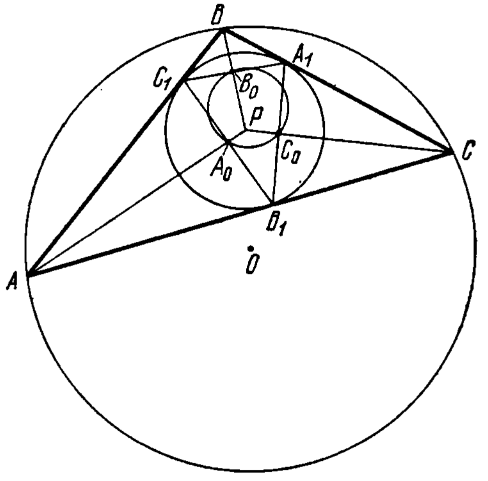

Fig. 74

circumscribed about triangle ABC, whose radius is R, goes into a circle of radius $r/2$ under the inversion $[P, r^2]$. Since the power of the inversion under consideration is $k = r^2$, and the power of the point P with respect to the circle (ABC) is equal to $\sigma = d^2 - R^2$, it follows that

$$\frac{r/2}{R} = \frac{r^2}{|d^2 - R^2|} = \frac{r^2}{R^2 - d^2},$$

whence

$$d^2 = R^2 - 2Rr$$

(Euler's formula).

Problem 7. N and S are two diametrically opposite points of a circle C; l is a straight line tangent to C at point S. From an arbitrary point O lying outside circle C but not lying on the tangent to C at N, we draw to circle C tangent lines OA and OB (A and B are the points of tangency). Let O', A' and B' be the projections from point N on the straight line l of points O, A and B. Prove that O' is the midpoint of segment $A'B'$ (Fig. 75).

Proof. Under the inversion $[N, NS^2]$, circle C goes into line l and circle K with centre O and radius $OA = OB$, which is orthogonal to circle C, goes into circle K', which is orthogonal to line l; hence, the centre of K' lies on l; on the other hand, the centre of K' also lies on line NO, and for this reason the projection O' of point O from point N on line l is the centre of K'. Hence, $A'B'$ is a diameter of K', and O' is the centre of K', and so $A'O' = O'B'$.

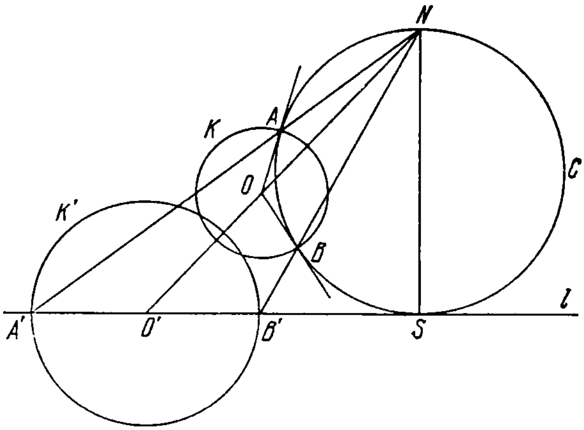

Fig. 75

Problem 8. Circles C_1 and C_2 do not have any points in common. Use inversion to transform them into two concentric circles and indicate how, under this inversion, the regions D_1, D_2, D_3 into which the circles C_1, C_2 divide the plane, are transformed (Fig. 76).

Solution. Construct some circle K that orthogonally cuts both circles C_1 and C_2. To do this, draw some circle M that cuts both circles C_1 and C_2 at the points P, Q and R, S respectively. Let O be the point of intersection of the straight lines PQ and RS (Fig. 77). Then the segments m of tangents drawn from point O to the circles C_1 and C_2 will be equal and, hence,

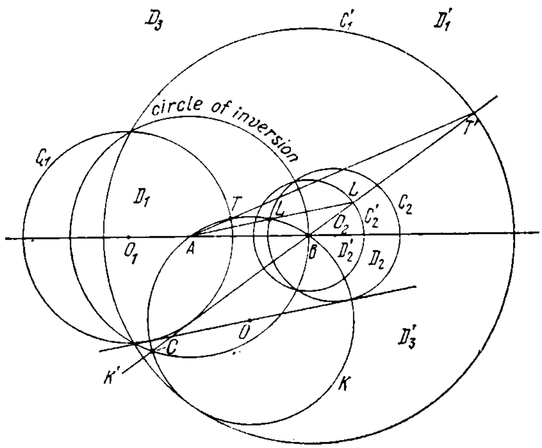

Fig. 76

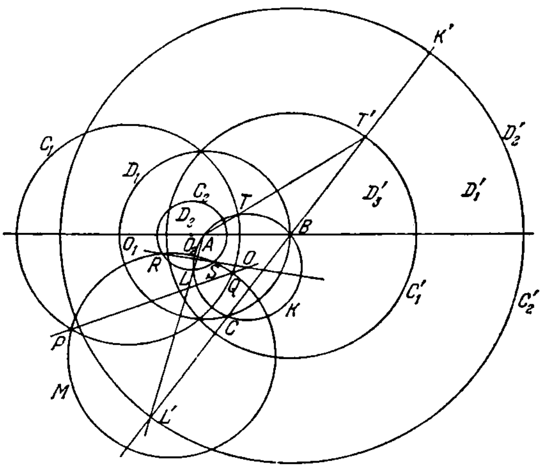

Fig. 77

the circle K with centre O and radius m will intersect both circles C_1 and C_2 orthogonally. Denote by A and B the points of intersection of K with the line of centres of C_1 and C_2. Then under the inversion $[A, AB^2]$, the circles C_1 and C_2 go into two concentric circles with centre B. Indeed, the circle K goes into the straight line K' that passes through point B and since K is orthogonal to the circles C_1 and C_2, it follows that line K' will be orthogonal to C_2' and C_1' into which circles C_1 and C_2 go; that is, the centres of C_1' and C_2' must lie on the straight line K'. But they also must lie on the straight line AB, hence, the centres of C_1' and C_2' coincide with point B.

Furthermore (see Fig. 76), since the centre A of inversion lies inside circle C_1, the region D_1 of points lying inside C_1 goes into the region D_1' of points lying outside C_1'. Since the centre A of inversion lies outside C_2, the region D_2 of points lying inside C_2 will go into the region D_2' of points lying inside C_2'. Hence the region D_3 of points lying outside C_1 and C_2 will go into a plane annulus bounded by the circles C_1' and C_2'.

Consider the case where the circle C_2 is put inside circle C_1 (see Fig. 77). In this case, the circle K that cuts C_1 and C_2 orthogonally goes into the line $K' = BC$, where C is the second point of intersection of K and the circle of inversion. If T and L are the points of intersection of K with C_1 and C_2 and if the straight lines TA and LA intersect K' in the points T' and L', then the images of C_1 and C_2 are the concentric circles C_1', C_2' with centre B and radii BT' and BL'. Furthermore, since point A lies inside C_2, it follows that the region D_2 of points lying inside C_2 goes, under the inversion $[A, AB^2]$, into the region D_2' of points lying outside C_2'. The region D_3 of points lying outside C_1 goes into the region D_3' of points lying inside C_1', and this means that the eccentric annulus D_1 bounded by C_1 and C_2 goes

into annulus D_1' that is bounded by the concentric circles C_1' and C_2' (Fig. 77).

Problem 9. Construct a circle tangent to three given circles C_1, C_2, C_3 *(Apollonian problem)*.

Solution. Consider only the case where each of the circles C_1, C_2, C_3 lies outside the other two. Let us perform an inversion under which C_1 and C_2 go into two concentric circles C_1' and C_2' (problem 8). Suppose that the pole A of this inversion $[A, AB^2]$ lies inside C_1, then $R_1' > R_2'$ where R_1' and R_2' are the respective radii of the circles C_1' and C_2'. Since C_3 lies outside C_1 and C_2, it follows that its image C_3' under the inversion $[A, AB^2]$ will lie inside the annulus formed by C_1' and C_2' since points lying outside C_1 go into points lying inside C_1', and points lying outside C_2 go into points lying outside C_2' (Fig. 78).

We split into two sets all the circles tangent to the concentric circles C_1' and C_2': circles of radius $(R_1' - R_2')/2$, each of which is externally tangent to C_2' and internally tangent to C_1', and the circles of radius $(R_1' + R_2')/2$, each of which is internally tangent to C_1' and C_2'. Only four circles K_1', K_2', K_3', K_4' belong to the first set, the first two of which are externally tangent to circle C_3', and K_3' and K_4' are internally tangent. To construct the circles K_1' and K_2' it suffices to construct their centres: these are the points of intersection of the circles $(O', (R_1' - R_2')/2)$ and $(O'', R_3' + (R_1' - R_2')/2)$, where O' and O'' are the respective centres of C_1' (or C_2') and C_3'. The centres of the circles K_3' and K_4' are the points of intersection of the circles $(O', (R_1' - R_2')/2)$ and $(O'', (R_1' - R_2')/2 - R_3')$.

Similarly, from among the circles of the second set that are internally tangent to the circles C_1' and C_2' there are only four circles S_1', S_2', S_3', S_4' that are externally tangent to circle C_3' (S_1' and S_2') and internally tangent

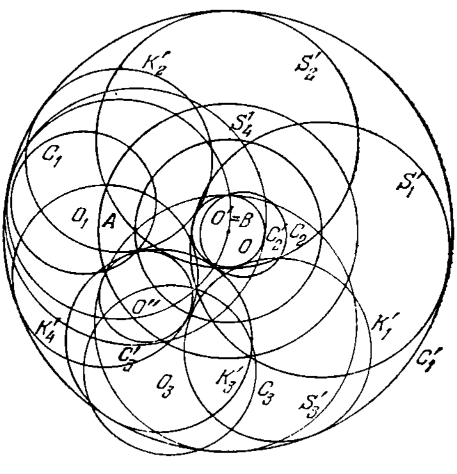

Fig. 78

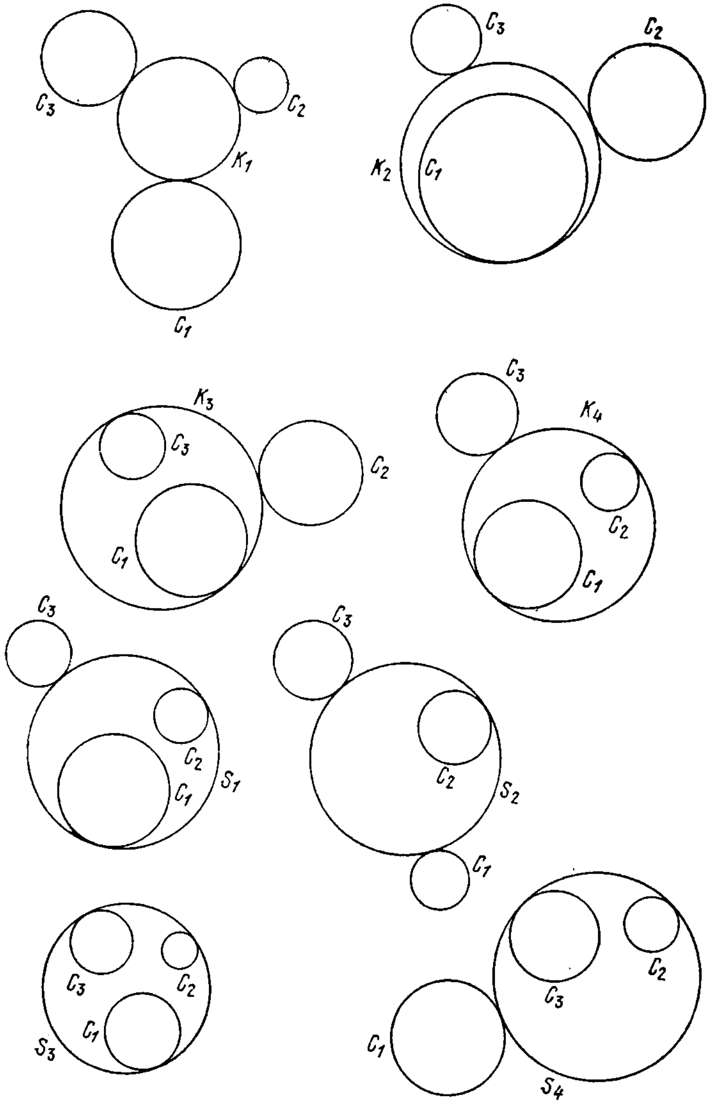

Fig. 79

to C_3' (S_3' and S_4'). The centres of the circles S_1' and S_2' are points of intersection of the circles (O', $(R_1' + R_2')/2$) and (O'', $(R_1' + R_2')/2 + R_3'$), and the centres of the circles S_3' and S_4' are the points of intersection of the circles (O', $(R_1' + R_2')/2$) and (O'', $(R_1' + R_2')\ /2 - R_3'$). All these eight circles K_1', K_2', K_3', K_4', S_1', S_2', S_3', S_4' are constructed in Fig. 78. Their images

$K_1, K_2, K_3, K_4, S_1, S_2, S_3, S_4$ under the inversion $[A, AB^2]$ will be tangent to the three given circles C_1, C_2, C_3. Thus, in the case at hand, the problem has eight solutions.

Figure 79 gives the positions of the circles K_i and S_i with respect to the circles C_1, C_2, C_3 on eight separate figures.

Problem 10. Prove that the Euler circle of $\triangle ABC$ is tangent to the circle (I) inscribed in that triangle and is tangent to the three circles (I_a), (I_b), (I_c) escribed in that triangle (the points of tangency are Φ_0, Φ_1, Φ_2, Φ_3 and are called *Feuerbach points*).

Solution. Let (I) and (I_a) be the respective circles (one inscribed in the given triangle and the other escribed in $\angle A$). Denote by R and S the points of tangency of (I) and (I_a) with side BC (Fig. 80). Then $BS = CR$ $(= p-c$, where p is the semiperimeter of $\triangle ABC)$. Let A', B', C' be the respective midpoints of the sides BC, CA, AB. Denote by A'', the orthogonal projection of point A on the side BC, and by Q the point of intersection of side BC with bisector II_a of $\angle BAC$. The set of points A, Q, I, I_a is an *harmonic* set of four points, that is, the double ratio of these four points is equal to -1. Hence the points A'', Q, R, S, which are the orthogonal projections of the points A, Q, I, I_a on line BC, will also be an harmonic set of four points. Point A' is the midpoint of segment RS (since A' is the midpoint of BC and $BS = CR$); hence, $A'Q \cdot A'A'' = A'R^2$. Let us consider the inversion $[A', A'R^2]$. Under this inversion, the circles (I) and (I_a) are invariant (since the circle of inversion is a circle with diameter RS and is orthogonal to both circles). From the relation $A'Q \cdot A'A'' = A'R^2$ it follows that under this inversion the point A'' goes into point Q.

On the other hand, point A'' (the foot of the altitude from A onto BC) lies on the Euler circle, the point A' (the midpoint of side BC) also lies on the Euler circle and, hence, under the inversion $[A', A'R^2]$ the Euler

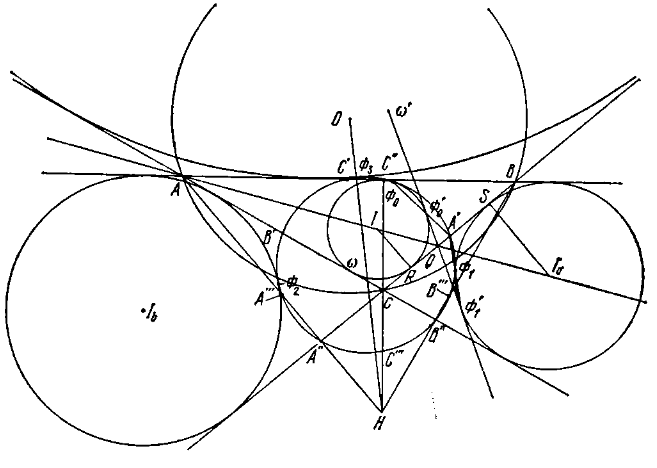

Fig. 80

circle goes into a straight line that passes through point Q and is antiparallel to the straight line $B'C'$ with respect to $\angle C'A'B'$* or into a straight line antiparallel to BC with respect to $\angle CAB$ (because $BC \parallel B'C'$, $CA \parallel C'A'$, $AB \parallel A'B'$). But the straight line that passes through point Q and is antiparallel to line BC (with respect to $\angle BAC$), is a straight line symmetric to line BC with respect to the bisector of $\angle BAC$, that is, the second common tangent ω' (interior) to the circles (I) and (I_a). Let Φ_0' and Φ_1' be points of tangency of ω' with (I) and (I_a). Under the inversion $[A', A'R^2]$, the straight line ω' is transformed into the Euler circle (ω), and the points Φ_0' and Φ_1' into the points Φ_0 and Φ_1 that lie on (I) and (I_a). In these points Φ_0 and Φ_1, the Euler circle is tangent to the inscribed circle (I) and to the escribed circle (I_a) because the straight line ω' is tangent to (I) and (I_a) in the points Φ_0' and Φ_1'; Φ_0 and Φ_1 are the points of intersection of the straight lines $A'\Phi_0'$ and $A'\Phi_1'$ with the Euler circle or the second points of intersection of these lines $A'\Phi_0'$ and $A'\Phi_1'$ with (I) and (I_a). Figure 80 depicts the construction of thirteen points $A', B', C', A'', B'', C'', A''', B''', C'''$, Φ_0, Φ_1, Φ_2, Φ_3 (all of them lie on the Euler circle).

Sec. 3. Mapping of regions under inversion

Problem 1. Suppose a circle C lies outside the circle K of the inversion $[O, r^2]$, where r is the radius of K. Let C' be the image of circle C under the inversion $[O, r^2]$. Then the set D_1 of all points lying inside C is mapped one to one onto the region D_1' of all points lying inside C'; the set D_2 of all points lying outside circles C and K is mapped one to one onto the set D_2' of all points lying inside K but outside C'. The connected region $D_2 \cup D_2'$ is invariant under the inversion $[O, r^2]$ (Fig. 81).

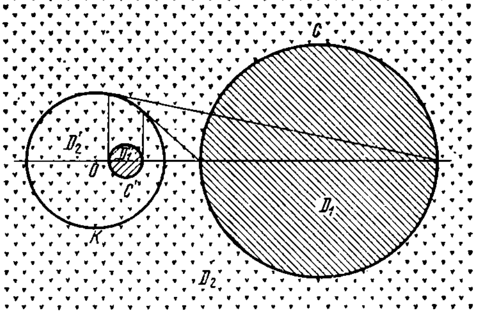

Fig. 81

* Straight lines that are symmetric with respect to the bisector of $\angle CAB$ are termed *antiparallel* with respect to that angle.

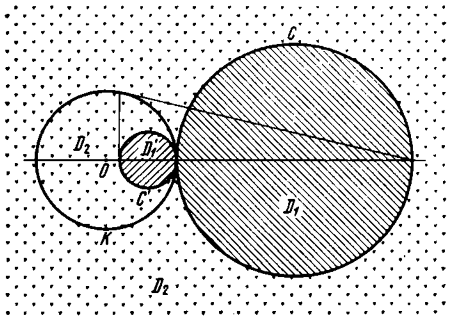

Fig. 82

Problem 2. A circle C is externally tangent to a circle K of the inversion $[O, r^2]$. Let C' be the inverse of C. The correspondence of the regions under the inversion $[O, r^2]$ is similar to that considered in problem 1 (Fig. 82).

Problem 3. A circle C cuts the circle K of the inversion $[O, r^2]$ but the centre O of the inversion lies outside C. The correspondence of the regions under the inversion is indicated in Fig. 83. The regions $D_2 \cup D_2'$, $D_3 \cup D_3'$, $D_4 \cup D_4'$, $D_6 \cup D_6'$ are simply connected and invariant under the inversion $[O, r^2]$. They are divided by the circle of inversion K into the regions D_2, D_2'; D_3, D_3'; D_4, D_4'; D_6, D_6' that pass into one another under the inversion at hand (Fig. 83).

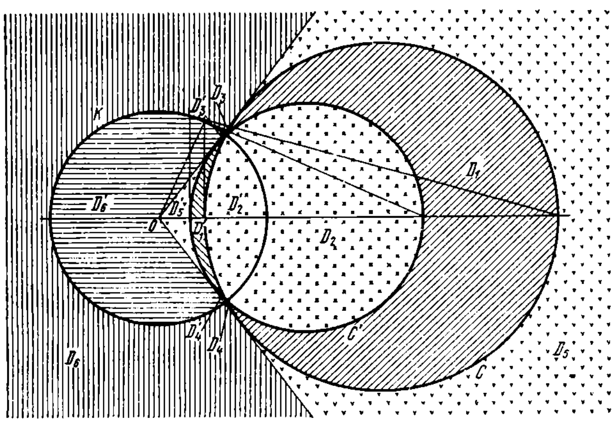

Fig. 83

Problem 4. A circle C passes through the centre of the inversion $[O, r^2]$ and cuts the circle of inversion K. Under the inversion $[O, r^2]$, the image C' of circle C is a straight line passing through the points of intersection of C and K. The correspondence of the regions under the inversion is shown in

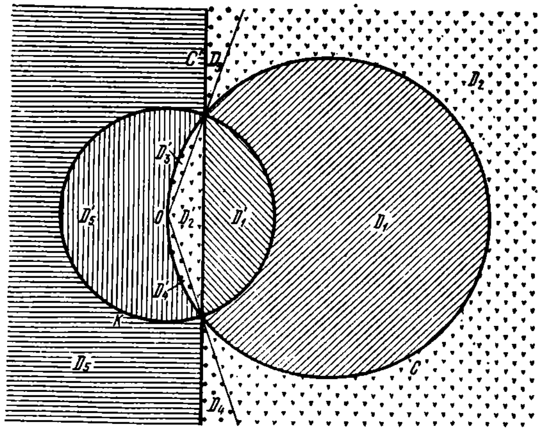

Fig. 84

Fig. 84. The regions $D_1 \cup D_1'$ and $D_5 \cup D_5'$ are simply connected and are invariant under the inversion $[O, r^2]$. The circle of inversion K divides them into the regions D_1 and D_1', D_5 and D_5', which go into one another under the inversion at hand (Fig. 84).

Problem 5. The circle of inversion K lies inside circle C. The image of the connected region D_1, which consists of points lying outside circle C is a simply connected region D_1', which consists of points lying inside circle C', where C' is the inverse of C with respect to circle K. The image of region D_2, which consists of points lying outside K, but inside C, is the region D_2', which consists of points lying inside K but outside C'. The connected region $D_2 \cup D_2'$ is invariant under the inversion under consideration (Fig. 85).

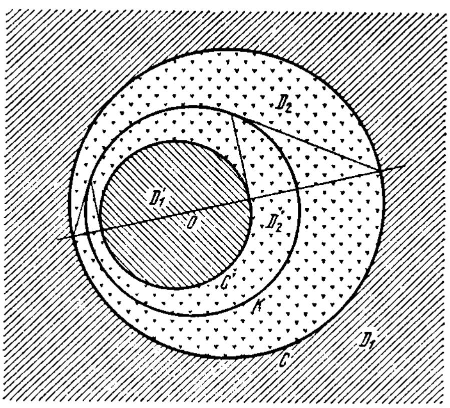

Fig. 85

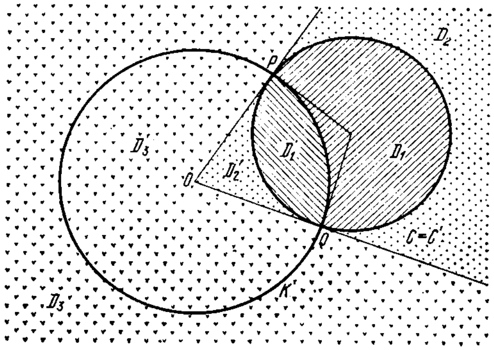

Fig. 86

Problem 6. A circle C intersects the circle of inversion K orthogonally. In this case the image C' of circle C (under the inversion with respect to circle K) coincides with the circle itself: $C = C'$. The region D_1, which consists of all points lying inside C but outside K goes into the region D_1', which consists of all points lying inside C and inside K. The region $D_1 \cup D_1'$, which consists of all points lying inside C is invariant under the inversion under consideration. Let P and Q be points in which the circles C and K intersect. Draw rays $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. The region D_2', which is bounded by arc $\overset{\frown}{PQ}$ of circle C and the radii OP and OQ of circle K goes into the region D_2, which is also bounded by arc $\overset{\frown}{PQ}$ of circle K (the complement of the first arc of this circle) and by the prolongations of the radii OP and OQ beyond points P and Q. Finally, the region D_3 consisting of points lying outside K and outside the angle POQ goes into region D_3', which consists of all points lying inside K but outside $\angle POQ$. The region $D_3 \cup D_3'$ is simply connected and invariant under the inversion with respect to circle K (Fig. 86).

Problem 7. A circle C passes through the centre O of the circle of inversion K and lies inside K. The image C' of circle C is a straight line that does not intersect K.

The region D_1, which consists of all points of the half-plane on the side of straight line C' that does not contain circle K, goes into a set D_1' of all points lying inside C. The region D_2, which consists of all points lying outside K and located in the half-plane (reckoned from line C') that contains K, goes into the set D_2' of all points lying inside circle K but outside circle C. The connected region $D_2 \cup D_2'$ is invariant under the inversion at hand (Fig. 87).

Problem 8. A circle C goes through the centre O of the circle of inversion K and is tangent to that circle. The image C' of circle C under the inversion with respect to circle K is the common tangent to C and K. The correspondence of regions D_1 and D_1', D_2 and D_2' is similar to the preceding problem. The region $D_2 \cup D_2'$ is simply connected and is invariant under the inversion with respect to the circle K (Fig. 88).

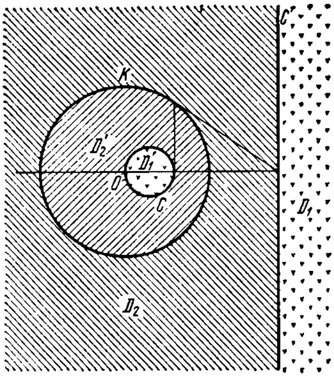

Fig. 87

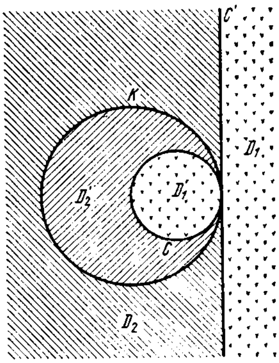

Fig. 88

Problem 9. A circle C intersects the circle of inversion K and the centre O of inversion (that is, the centre of circle K) lies inside C. The correspondence of regions D_1 and D_1', D_2 and D_2', D_3 and D_3', D_4 and D_4' is shown in Fig. 89. The regions $D_1 \cup D_1'$, and $D_4 \cup D_4'$ are simply connected and are invariant under the inversion with respect to circle K.

Problem 10. Given three equal circles C_1, C_2, C_3 that pass through one point O and intersect at angles of $\pi/3$. Take point O as the pole of inversion and take the circle of inversion K so that it intersects C_1, C_2, C_3 and so that the points of intersection lie inside K. The images C_1', C_2', C_3' of the given circles C_1, C_2, C_3 are straight lines passing through the points of intersection of circle K with each of the circles C_1, C_2, C_3. The given circles C_1, C_2, C_3, their images C_1', C_2', C_3' (under the inversion with respect to the circle K), and the circle K itself divide the plane into 24 regions. The correspondence of regions under the inversion of the plane with respect to circle K is shown in Fig. 90. The regions $D_7 \cup D_7'$, $D_8 \cup D_8'$, $D_9 \cup D_9'$, $D_{10} \cup D_{10}'$, $D_{11} \cup$ $\cup$ D_{11}', $D_{12} \cup D_{12}'$ are simply connected and invariant under the inver-

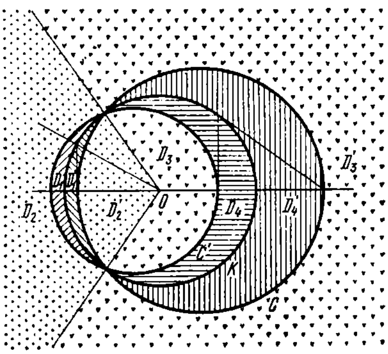

Fig. 89

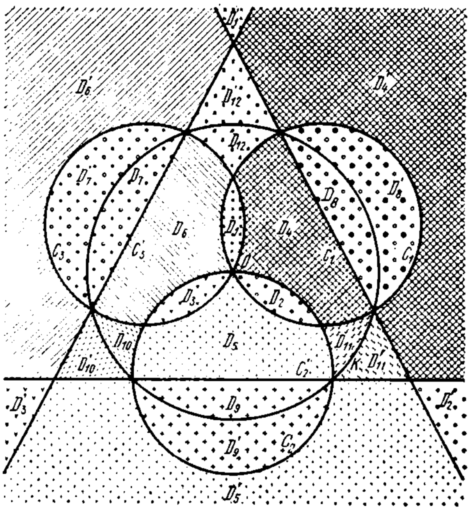

Fig. 90

sion at hand. The regions D_k and D'_k ($k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$) go into one another under this inversion.

Problem 11. Consider the cardioid

$$\rho = 1 + \cos\varphi,$$

the equation of which is given in polar coordinates. Under the inversion $[O, 1]$, where O is the pole of the polar system of coordinates, the image of the cardioid is a parabola whose equation in that same polar coordinate system is of the form

$$\rho = \frac{1}{1 + \cos\varphi}.$$

If a rectangular Cartesian system of coordinates is introduced and the polar axis is taken as the x-axis, then the last equation becomes

$$y^2 = 1 - 2x.$$

Figure 91 depicts a parabola constructed on the basis of this equation. The cardioid $\rho = 1 + \cos\varphi$, the parabola $y^2 = 1 - 2x$, and the circle of inversion K divide the plane into six regions. The correspondence of these regions under the inversion with respect to circle K is depicted in Fig. 91. The regions $D_1 \cup D'_1$ and $D_3 \cup D'_3$ are simply connected and invariant under the inversion with respect to circle K ($D_1 \rightleftarrows D'_1$, $D_3 \rightleftarrows D'_3$). Region D_2, which is bounded by a part of the cardioid and an arc of the parabola, goes into region D'_2, which is made up of points lying outside the cardioid and the parabola. Conversely, under the inversion at hand, region D'_2 goes into region D_2.

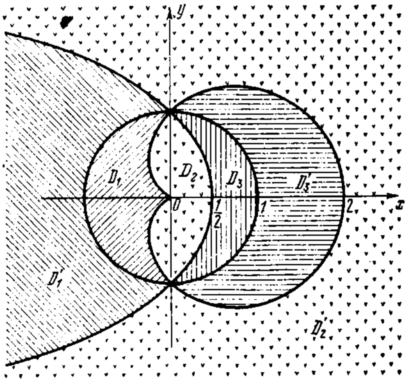

Fig. 9

Problem 12. We consider Pascal's limaçon, the equation of which in polar coordinates is of the form

$$\rho = m + n\cos\ \varphi.$$

We assume that $0 < n < m$. Then we have $\rho > 0$ for all values of φ, and, hence, the limaçon considered here is a closed curve without self-intersections, inside which curve is the pole O of the polar system of coordinates. Under the inversion $[O, 1]$, where O is the pole of the polar system of coordinates, the limaçon goes into a curve, whose equation in that same polar system is of the form

$$\rho = \frac{1}{m + n\cos\varphi}$$

or

$$\rho = \frac{p}{1 + e\cos\varphi},$$

where $p = 1/m$, $e = n/m$. Since from the conditions $0 < n < m$ it follows that $0 < e < 1$, the last equation is the equation of an ellipse for which O is one of the foci, p is a parameter, and e is the eccentricity.

For example, take $m = 3/2$, $n = 1$. Then the equation of Pascal's limaçon is

$$\rho = \frac{3}{2} + \cos\varphi.$$

and the equation of the ellipse, which is the image of this limaçon under the inversion $[O, 1]$, is

$$\rho = \frac{1}{\frac{3}{2} + \cos\varphi}.$$

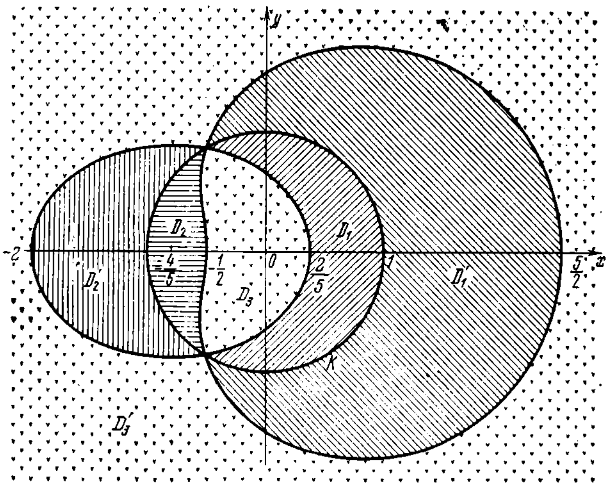

Fig. 92

If a rectangular Cartesian system of coordinates is introduced, and the polar axis is taken as the x-axis, then the last equation becomes

$$\frac{\left(x+\frac{4}{5}\right)^2}{\frac{36}{25}}+\frac{y^2}{\frac{4}{5}}=1,$$

which is an ellipse with centre at the point $(-4/5, 0)$ and with semi-axes equal to $a = 6/5$, $b=2/\sqrt{5}$. Figure 92 depicts the limaçon under consideration and the ellipse, which intersects the limaçon $\rho = 1.5 + \cos\varphi$ for values of φ that satisfy the equation

$$\frac{3}{2}+\cos\varphi=\frac{1}{\frac{3}{2}+\cos\varphi},$$

whose $\cos\varphi = -\frac{1}{2}$; and, thus, assuming that $0 \leqslant \varphi < 2\pi$, we have $\varphi = 2\pi/3$, $\varphi = 4\pi/3$ (two points of intersection). For these values of φ we have $\rho = 1$ and, hence, the circle of inversion K passes through the points of intersection of the limaçon and the ellipse.

The limaçon, the ellipse, and the circle of inversion K divide the plane into six regions. The correspondence of these regions is shown in Fig. 92. The regions $D_1 \cup D_1'$ and $D_2 \cup D_2'$ are simply connected and invariant under inversion, and under the inversion at hand they pass into one anoth-

er: $D_1 \rightleftarrows D_1'$, $D_2 \rightleftarrows D_2'$. The region D_3, which is bounded by an arc of the limaçon and an arc of the ellipse, contains the centre O of inversion (the pole of the polar coordinate system), goes into the region D_3', which consists of all points lying outside the ellipse and the limaçon. Conversely, under the inversion $[O, 1]$, the region D_3' goes into the region D_3.

Problem 13. Let us consider Pascal's limaçon once again in polar coordinates; it is given by the equation

$$\rho = m + n \cos \varphi,$$

but it is now assumed that $0 < m < n$. The equation $\rho = 0$ now has the solution

$$\cos \varphi = -m/n$$

and if we assume $0 \leqslant \varphi < 2\pi$, then the last equation yields two values for φ:

$$\varphi = \arccos(-m/n), \quad \varphi = 2\pi - \arccos(-m/n).$$

Therefore, when $0 < m < n$ the limaçon passes through the pole twice. In the case at hand, this is a curve with self-intersection; it forms a loop inside the rest of the curve. Under the inversion $[O, 1]$, where O is the pole of the polar system of coordinates, the limaçon goes into a curve whose equation in the same polar system of coordinates is of the form

$$\rho = \frac{1}{m + n \cos \varphi}$$

or

$$\rho = \frac{p}{1 + e \cos \varphi},$$

where $p = 1/m$, $e = n/m$. Since we now have $e > 1$, the curve specified by this equation is a hyperbola with eccentricity e and parameter p (half the focal chord).

Let us now take, for example, $m = 1$, $n = 2$ (Fig. 93). The equations of the limaçon and its image (hyperbola) under the inversion $[O, 1]$ are

$$\rho = 1 + 2 \cos \varphi, \quad \rho = \frac{1}{1 + 2 \cos \varphi}.$$

Transform the equation of the hyperbola by introducing a rectangular Cartesian system of coordinates in which the x-axis is the polar axis. We then have

$$\frac{\left(x - \frac{2}{3}\right)^2}{\frac{1}{9}} - \frac{y^2}{\frac{1}{3}} = 1.$$

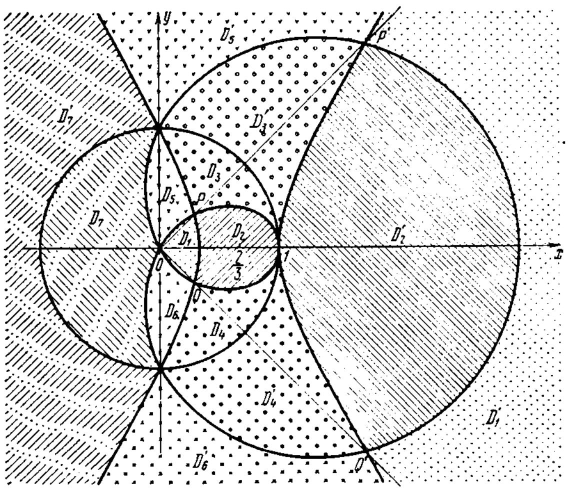

Fig. 93

This is a hyperbola with centre (2/3, 0) and semi-axes $a = 1/3$, $b = 1/\sqrt{3}$. Since $b/a = \sqrt{3}$, it follows that the asymptotes of the hyperbola are inclined to the x-axis at angles of $\pm\pi/3$. When $\varphi = \pm 2\pi/3$ in the equation of the limaçon $\rho = 1 + 2\cos\varphi$, the radius vector ρ vanishes. Therefore the asymptotes of the hyperbola are parallel to tangents to the loop of the limaçon at the coordinate origin. The limaçon, the hyperbola, and the circle of inversion K divide the plane into 14 regions. The correspondence $D_k \rightleftarrows D_k'$ of these regions is depicted in Fig. 93. The regions $D_3 \cup D_3'$, $D_4 \cup D_4'$, $D_7 \cup D_7'$ are simply connected and are invariant under our inversion; also the circle of inversion divides them into the regions D_3 and D_3', D_4 and D_4', D_7 and D_7', which go into one another under the inversion. The limaçon and the hyperbola intersect in seven points; three of them lie also on the circle of inversion; the other four P, P', Q, Q' do not lie on the circle of inversion and correspond to one another ($P \rightleftarrows P'$, $Q \rightleftarrows Q'$) under inversion (symmetry with respect to the circle K). Using the straight lines OPP' and OQQ', we can partition the regions D_1 and D_1' into three regions: $\overset{*}{D}_1$, $\overset{**}{D}_1$, $\overset{***}{D}_1$ and $\overset{*}{D}{}_1'$, $\overset{**}{D}{}_1'$, $\overset{***}{D}{}_1'$, which will correspond to one another under the inversion at hand. These same lines divide the regions D_3, D_3' and D_4, D_4' into two parts each, and they correspond to each other under the inversion $[O, 1]$.

Problem 14. The equation of the cissoid of Diocles in polar coordinates may be written as

$$\rho = \frac{2\sin^2\varphi}{\cos\varphi}. \tag{2}$$

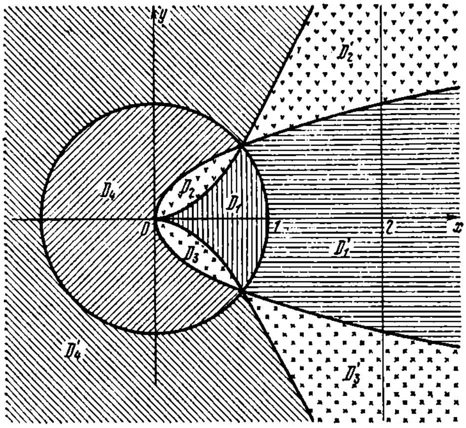

Fig. 94

The straight line $x = 2$ is a vertical asymptote of the cissoid because

$$\lim_{\varphi \to \frac{\pi}{2} - 0} \rho = +\infty;$$

and from the equation (2) it follows that

$$\lim_{\varphi \to \frac{\pi}{2} - 0} x = \lim_{\varphi \to \frac{\pi}{2} - 0} \rho \cos \varphi = \lim_{\varphi \to \frac{\pi}{2} - 0} (2 \sin^2 \varphi) = 2.$$

Under the inversion $[O, 1]$ (O is the pole of the polar system of coordinates), the cissoid is transformed into a curve whose equation in the same polar system is of the form

$$\rho = \frac{\cos \varphi}{2 \sin^2 \varphi}$$

or

$$2y^2 = x.$$

This is a parabola for which the polar axis serves as the axis. The circle of inversion K passes through two points in which the cissoid and parabola intersect. (Note that the circle of inversion does not intersect the asymptote $x = 2$ of the cissoid.) The cissoid, the parabola, and the circle of inversion divide the plane into eight regions D_k, D_k' ($k = 1, 2, 3, 4$), which pass into one another under the inversion at hand: $D_k \rightleftarrows D_k'$.

The regions $D_1 \cup D_1'$ and $D_4 \cup D_4^1$ are simply connected and are invariant under the inversion $[O, 1]$ (Fig. 94).

Sec. 4. Mechanical inversors: the Peaucellier cell and the Hart cell

Suppose $MPM'Q$ is a rhombus. The point O is a point equidistant from points P and Q. Also suppose the rhombus is hinged, point O being fixed and joined by hinges with P and Q (the *Peaucellier cell*). Then if M describes some curve L, the point M' describes a curve L', which is the image of L under the inversion $[O, OP^2 - PM^2]$ (Fig. 95).

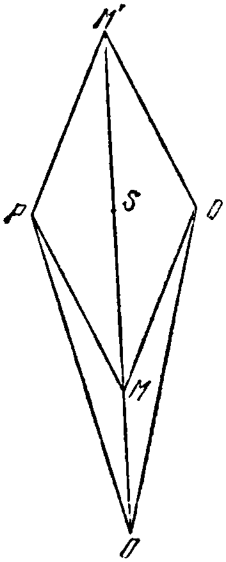

Fig. 95

Proof. The points M, M' and O are at equal distances from the points P and Q, and for that reason lie on the midperpendicular of segment PQ. Let us construct the circle (P, PM). The product $\overline{OM}\cdot\overline{OM'}$ is equal to $OP^2 - PM^2$. Indeed, let S be the midpoint of segment PQ; then

$$\begin{aligned}\overline{OM}\cdot\overline{OM'} &= (\overline{OS}+\overline{SM})(\overline{OS}+\overline{SM'})\\ &= (\overline{OS}+\overline{SM})(\overline{OS}-\overline{SM})\\ &= OS^2 - SM^2 = OP^2 - SP^2 - (MP^2 - SP^2)\\ &= OP^2 - PM^2.\end{aligned}$$

Note, in particular, that if point M describes a circle passing through point O, then point M' describes a straight line. Thus, the Peaucellier cell makes it possible, mechanically, to transform circular motion into rectilinear motion.

Let $ABCD$ be an antiparallelogram (that is, $ABDC$ is an isosceles trapezoid, BD and AC are its parallel sides, AD and BC the diagonals). Fix a point O on segment AB and a point M on segment AD, and on segment BC fix a point M' so that the points O, M, and M' belong to a single straight line parallel to $BD \| AC$. The points A, B, C, D, O have hinges; the point O is fixed (this is *Hart's cell*). If under these conditions we deform the antiparallelogram $ABCD$, then the product $\overline{OM}\cdot\overline{OM'}$ will remain constant, that is, if point M describes a curve L, then point M' will describe a curve obtained from L by an inversion with the pole O (Fig. 96).

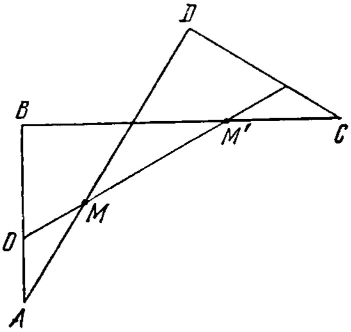

Fig. 96

Proof. If the antiparallelogram $ABCD$ is hinged, then, as will readily be seen, when it is deformed it remains an antiparallelogram (this stems from the fact that an antiparallelogram is characterized by the equalities $AB = CD$, and $AD =$

$= BC$). On the other hand, the straight lines OM and OM', which are parallel to the bases BD and AC of the trapezoid in the original position of the figure, will all the time remain parallel to them. Indeed,

$$\frac{AM}{AD} = \frac{AO}{AB}, \quad \frac{BM'}{BC} = \frac{BO}{AB}.$$

When the antiparallelogram $ABCD$ is deformed, these proportions are preserved. In particular, from what has been said, the points O, M and M' remain on one straight line. Furthermore, from similar triangles we find

$$\frac{OM}{BD} = \frac{AO}{AB}, \quad \frac{OM'}{OB} = \frac{AC}{AB},$$

whence

$$\overline{OM}\cdot\overline{OM'} = \frac{OA\cdot OB}{AB^2}\cdot BD\cdot AC.$$

But a circle can be circumscribed about the trapezoid $ABCD$, and so

$$AB^2 + BD\cdot AC = AD^2$$

And, hence,

$$BD\cdot AC = AD^2 - AB^2.$$

Finally, we get

$$\overline{OM}\cdot\overline{OM'} = \frac{OA\cdot OB}{AB^2}(AD^2 - AB^2).$$

Therefore point M' is obtained from point M under the inversion $[O, k]$, where

$$k = \frac{OA\cdot OB(AD^2 - AB^2)}{AB^2}.$$

Sec. 5. The geometry of Mascheroni

We will assume that a straight line is specified if two distinct points of the line are fixed in the plane. A circle is regarded as specified if the following are given: the centre and also a point lying on the circumference of the circle, or three points on the circumference. We will denote a circle with centre O and radius OM as (O, OM). A circle specified by three points A, B, C lying on the circumference will be symbolized as (ABC).

We now consider the solution of a number of basic problems involving construction with a compass alone. Most of the solutions involve the use of inversion.

Starting mainly with these constructions, which were carried out without the aid of inversion, the Danish mathematician G. Mohr (17th century)

succeeded in proving that with a compass alone it is possible to perform *all* constructions that can be performed with compass and straight edge.

This proposition was independently proved at the end of the 18th century by the Italian mathematician L. Mascheroni, and it was precisely Mascheroni's work that became known in Europe (Mohr's book had gone unnoticed). For this reason, constructions peformed *with only a compass* are frequently associated with the name of Mascheroni (the geometry of Mascheroni, Mascheroni constructions and so forth).

We now consider several construction problems involving only a compass.

Problem 1. A line segment AB is specified by its endpoints. Construct, on the extension of segment AB, beyond point B, a point C such that $AC = n \cdot AB$, where n is a natural number.

Solution. We construct the circles (A, AB) and (B, BA). Let P be one of the points of their intersection. Construct a circle (P, PA); let Q be the second point of intersection of this circle with (B, BA). We construct the circle (Q, QB); let R be the second point of intersection of this circle with (B, BA). The point R lies on the extension of segment AB beyond B, and $AB = BR$ (Fig. 97).

True enough, the indicated construction yields the vertex R of a regular hexagon; R is opposite vertex A and the hexagon is inscribed in (B, BA). Continuing similar constructions, we can construct point C that lies on the extension of segment AB beyond point B and such that $AC = n \cdot AB$, where n is any natural number.

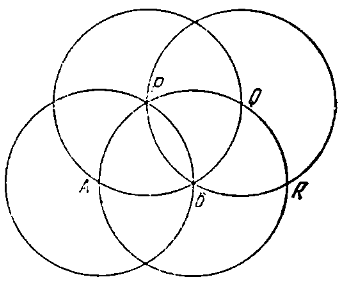

Fig. 97

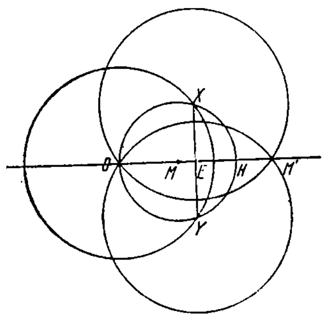

Fig. 98

Problem 2. Construct with compass alone the image M of a point M' under the inversion $[O, r^2]$.

Solution. 1°. $OM > r/2$. If point M lies on the circle of inversion, then its image M' coincides with itself. Therefore, let us assume that point M lies either inside the circle of inversion (Fig. 98) or outside it (Fig. 99). In both cases we construct the circle (M, MO). Let X and Y be the points of intersection of this circle with the circle of inversion. We construct the circles (X, XO) and (Y, YO); the second point of intersection of these circles is the point M'. Indeed, since the points X and Y are symmetric about the

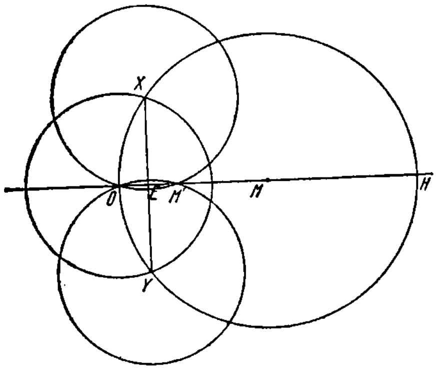

Fig. 99

straight line MO, and the circles (X, XO) and (Y, YO) are congruent, it follows that the second point M' of their intersection lies on the radial line $\overrightarrow{OM}$. Furthermore, if H is a point diametrically opposite point O on the circle (M, MO), and E is the point of intersection of XY and OM (E is the midpoint of segment OM'; neither point H nor point E need be constructed, they are introduced merely to aid the proof), then

$$r^2 = OX^2 = OH \cdot OE = OM \cdot OM'.$$

2°. $OM < r/2$. In this case (M, MO) does not intersect the circle of inversion. We construct on the radial line $\overrightarrow{OM}$ a point N such that $n \cdot \overrightarrow{OM} = \overrightarrow{ON}$ and such that $ON > r/2$. Construct point N' obtained by the inversion $[O, r^2]$ from point N (case 1°) and then construct on the radial line $\overrightarrow{ON'}$ a point M' such that $\overrightarrow{OM'} = n \cdot \overrightarrow{ON'}$; then

$$\overline{OM} \cdot \overline{OM'} = \frac{\overline{ON}}{n} \cdot n\overline{ON'} = \overline{ON} \cdot \overline{ON'} = r^2.$$

From the solution of this problem there follows the possibility of constructing with compass alone a point C lying on segment AB and such that $n \cdot AC = AB$. Indeed, construct on the extension of segment AB, beyond point B, a point C' such that $AC' = n \cdot AB$ (problem 1). Let C be the image of point C' under the inversion $[A, AB^2]$ (problem 2). Point C is the desired one. Indeed,

$$\overline{AC} \cdot \overline{AC'} = AB^2, \; AC \cdot nAB = AB^2, \; n \cdot AC = AB.$$

Problem 3. Using only a compass, construct a circle C' as the image, under the inversion $[O, r^2]$, of the straight line C that does not pass through the pole O of the inversion. The straight line is assumed to be given by two distinct points X and Y lying on it.

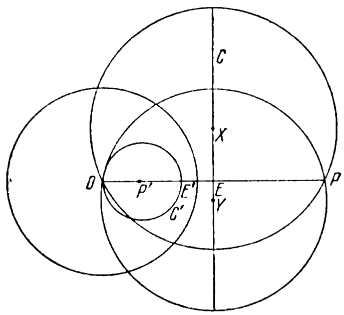

Fig. 100

Solution. The circle C' passes through point O. Let P be the second point of intersection of the circles (X, XO) and (Y, YO). Then P is symmetric to point O with respect to the line XY. Construct (with compass alone) the image P' of point P under the inversion $[O, r^2]$. The desired circle C' is the circle $(P', P'O)$ (Fig. 100). Indeed, if E is the point of intersection of the straight lines XY and OP [E is the midpoint of OP and E' is a point diammetrically opposite to point O on circle $(P', P'O)$] then

$$r^2 = OP \cdot OP' = OE \cdot OE'.$$

Problem 4. Using only a compass, construct the point of intersection of two straight lines, one of which is given by the points A and B, the other by the points C and D.

Solution. Construct an arbitrary circle S in the plane. Construct (with compass alone) the circles K_1 and K_2, which are inverses of the given lines with respect to the circle S. Let M' be the point of intersection of the circles K_1 and K_2. Construct (with compass alone) the inverse M of point M' with respect to the circle S. The point M is just the point of intersection of the given straight lines.

Problem 5. Construct (with compass alone) the centre of a circle specified by three points A, B, C lying on that circle (the circle itself is not drawn).

Solution. Let P and Q be the points of intersection of the circles (A, AB) and (B, BA). The straight line PQ is the midperpendicular of segment AB. Let L and M be the points of intersection of (B, BC) and (C, CB). The line LM is the midperpendicular of segment BC. The centre O of the circle (ABC) is the point of intersection of the straight lines PQ and LM (problem 4).

Problem 6. On a straight line given by two points A and B, lay off from point B segments BC_1 and BC_2 equal to the given segment PQ.

Solution. Construct a circle (B, PQ). The problem reduces to seeking the points of intersection of that circle with line AB. Construct (with compass alone) the images K_1 and K_2 of line AB and the circle (B, PQ) under inversion with an arbitrary circle of inversion. Let C_1' and C_2' be the points of intersection of the circles K_1 and K_2. The inverses C_1 and C_2 of points C_1' and C_2' are the desired points.

Problem 7. Take three given line segments PQ, MN, RS and construct a fourth, proportional to them, that is, such that

$$PQ \cdot MN = RS \cdot XY.$$

Solution. 1°. $PQ \neq MN$. Construct circles (O, PQ) and (O, MN), where O is an arbitrary point in the plane. On (O, PQ) take an arbitrary point A and construct (with compass alone) a point B in which the straight line OA intersects (O, MN) (take the point B which lies on the radial line $\overrightarrow{OA}$). Construct a circle (O, RS) and on it take an arbitrary point C. Construct a circle (ABC) (problem 5). Let D be the second point of intersection of this circle with line OC (it is constructed with a compass alone). Then

$$OA \cdot OB = OC \cdot OD$$

or

$$PQ \cdot MN = RS \cdot OD;$$

OD is the desired line segment.

2°. $PQ = MN$. In this case we consider segments PQ and $2MN$ (segment $2MN$ is constructed with compass alone). Construct (with compass alone) the segment X_1Y_1 such that

$$PQ \cdot 2MN = RS \cdot X_1Y_1 \qquad \text{(case 1°)}.$$

The desired segment $XY = \dfrac{X_1Y_1}{2}$ is constructed by compass alone (see remark at the end of the solution of problem 2).

Problem 8. Given in the plane two points A and B. Construct (with compass alone) a point C such that the angle ABC is equal to 90°.

Solution. On the straight line AB we construct (with compass alone) the line segment $BD = AB$. Let C be any one of the points of intersection of the circles (A, AD) and (D, DA). The point C is the desired point.

Problem 9. Given a segment AB with points A and B. Let n be an arbitrary natural number not the square of any natural number. Construct the segment $AB\sqrt{n}$.

Solution. Construct (with compass alone) a point C such that $\angle ABC = 90°$. Find the point P of intersection of BC with the circle (B, BA). Then $AP = AB\sqrt{2}$. Then construct (with compass alone) a point D such that $\angle APD = 90°$ and construct the point Q of intersection of the straight line PD with the circle (P, AB). Then $PQ = AB\sqrt{3}$ and so on.

From the results of this problem it follows that it is possible with compass alone to construct on a circle (specified by three points) the vertices of a regular triangle, a square, a regular pentagon and hexagon.

Sec. 6. Inversion of space

The inversion $[O, k]$ of space (more precisely, of *spherical space* with a single ideal point, or point at infinity, adjoined) is defined in the same way as the inversion of a plane. The inversions $[O, k]$ of space, like those

of a plane, separate into positive inversions ($k > 0$) and negative inversions ($k < 0$) depending on the sign of the power k of inversion.

Under a positive inversion $[O, k]$, a sphere S_0 with centre O and radius $\sqrt{k}$ (termed the *sphere of inversion*) consists of invariant points.

The set of points lying outside the sphere S_0 is mapped into the set of interior points of S_0, and the set of all points lying inside S_0 is mapped onto the set of all points lying outside S_0.

If $k < 0$, then a sphere with centre O and radius $\sqrt{|k|}$ is invariant, and each point of it is mapped into a point diametrically opposite on the sphere S_0.

If the inversion $[O, k]$ is positive, then all spheres intersecting the sphere S_0 of inversion orthogonally, and only such spheres, are invariant.

The inverse of a sphere not passing through the centre of inversion is a sphere.

The tangency of spheres and the tangency of a sphere and a plane are preserved under inversion.

If a sphere S with centre Q passes through the centre O of inversion, then its image, under the inversion $[O, k]$, is a plane orthogonal to the straight line OQ, which plane does not pass through the pole of inversion, and, conversely, a plane that does not pass through O goes into a sphere that passes through O.

The inversion of space is a conformal transformation, that is, under the inversion, angles between tangent planes in the points of intersection of two spheres are preserved.

A circle that does not pass through the pole of inversion goes into a circle. Indeed, every such circle may be regarded as the line of intersection of two spheres not passing through the pole of inversion.

Under the inversion of space, angles are preserved between intersecting circles (which, generally speaking, lie in distinct planes). Indeed, let a and b be tangents to the circles C_1 and C_2 in the point of their intersection M. It suffices to prove that under the inversion $[O, k]$ the angle is preserved between the straight lines a and b. We consider only that case where the straight lines a and b do not pass through the pole of inversion O. Let a^* and b^* be straight lines that are respectively parallel to straight lines a and b and that pass through the pole of inversion O. Under the inversion $[O, k]$, straight lines a and b go into the circles a' and b' that are respectively tangent to the straight lines a^* and b^* at point O. Therefore, the angle between the circles a' and b' is equal to the angle between the straight lines a^* and b^*, that is, it is equal to the angle between a and b because $a^* \parallel a$, $b^* \parallel b$.

* * *

We now introduce a number of geometric constructions and analytic derivations associated with stereographic projection in connection with the use of this mapping of a sphere on a plane when making maps

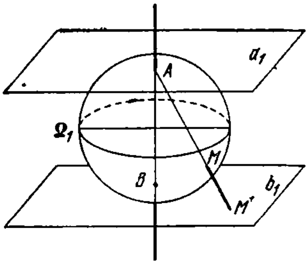

Fig. 101

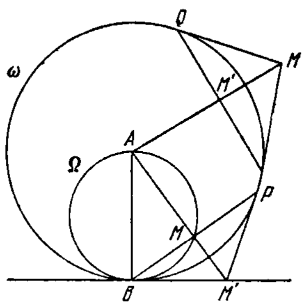

Fig. 102

of the earth. We dwell solely on the geometric construction of the depiction of the network of meridians and parallels and on the analytic investigation of the nature of their distribution (under stereographic projections).

Let us recall the basic definitions and properties of a stereographic projection in connection with the inversion transformation.

Let A and B be the ends of some diameter AB of a sphere Ω_1; a_1 and b_1 are planes tangent to Ω_1 at the points A and B. Let M be an arbitrary point of Ω_1 distinct from point A. Denote by M' the point in which the straight line AM intersects the plane b_1. The correspondence under which point M is associated with point M' is termed the *stereographic projection* of the sphere Ω_1 on the plane b_1 (Fig. 101).

Let us consider a section Ω of the sphere Ω_1 cut by the plane ABM (Fig. 102). From the similarity of $\triangle ABM$ and $\triangle ABM'$ we find

$$AM \cdot AM' = AB^2.$$

From this relation it follows that the image M' of point M under the stereographic projection of the sphere Ω_1 on the plane b_1 is also the image of point M under the inversion $I = [A, AB^2]$ with pole A and the power of inversion equal to AB^2; in other words, point M' is the image of point M under an inversion with the sphere of inversion ω_1 the centre of which is point A and radius AB. To construct the image M' of point M under the inversion I it is first convenient to construct sections Ω and ω of the spheres Ω_1 and ω_1 by the plane ABM. Then join point M with point A and draw a straight line passing through point M perpendicular to straight line AM. Let P be the point in which this line intersects the circle ω; then the tangent to ω at the point P will cut line AM at point M'. This construction is true of any point M lying inside the circle ω. If point M lies on ω, then its image is the point M itself. If M lies outside ω, then to construct the image M' under the inversion with respect to circles ω, we construct the tangent MQ drawn from point M to ω (Q is the point of tangency); the orthogonal projection M' of point Q on line AM is precisely the image of point M under the inversion I. All these constructions can be carried out in any section Ω of the sphere Ω_1 by a plane

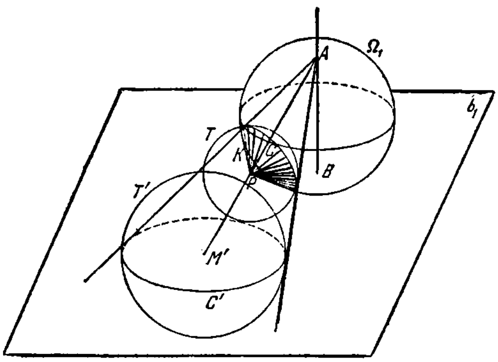

Fig. 103

passing through the straight line AB. Incidentally, they can also be carried out in space by drawing the appropriate tangents not to the circles Ω and ω, but to the spheres Ω_1 and ω_1.

From the foregoing we arrive at the following important conclusion: on the set of all points of Ω_1 (with the exception of point A), a stereographic projection coincides with an inversion, the pole of which is the centre of projection A, and the power of the inversion is equal to the square of the diameter of Ω_1; in other words, a stereographic projection on the set of all points of Ω_1 (with the exception of point A) coincides with an inversion with the sphere of inversion ω_1, the centre of which is point A, and the radius is equal to the diameter of Ω_1.

From this it follows that the geometric properties of a stereographic projection may be obtained from the familiar properties of inversion, namely: under an inversion (and, hence, under a stereographic projection) the circles that lie on the sphere Ω_1 and do not pass through point A go into circles (lying in the plane b_1). Since inversion is a conformal transformation (that is, angles are preserved), it follows that a stereographic projection has the same property, so that using a stereographic projection makes it possible to construct a conformal map of the globe (it is not, however, possible to construct a map of the earth on which distances are preserved).

Let us now see how one constructs the centre of circle C', into which circle C, which lies on the sphere Ω_1 and does not pass through point A, is mapped. Let us first suppose that C is not a great circle of Ω_1. We construct a cone K tangent to Ω_1 along the circle C. Let P be the vertex of the cone. We construct a sphere T with centre P that passes through circle C (Fig. 103). This sphere intersects Ω_1 orthogonally. Under the inversion $I = [A, AB^2]$, the sphere Ω_1 goes into the plane b_1, and the sphere T goes into the sphere T', which will intersect the plane b_1 orthogonally. From this it follows that the centre M' of T' must lie on the plane b_1. But, on the other hand, the centre of sphere T' must also lie on line AP

since sphere T' is the image of sphere T under the inversion $[A, AB^2]$. Thus, the centre M' of sphere T' is a point in which the straight line AP intersects the plane b_1. Furthermore, the circle C is the line of intersection of the spheres Ω_1 and T, hence the image C' of circle C (both under the inversion I and under the stereographic projection under consideration) is the intersection of the sphere T' and the plane b_1. But since the centre M' of the sphere T' lies in the plane b_1, it follows that C' is a great circle of sphere T', and therefore the centre of circle C' coincides with the centre M' of sphere T', that is, it is the projection M' on plane b_1 of the vertex P of the cone K that is tangent to the sphere Ω_1 along the circle C.

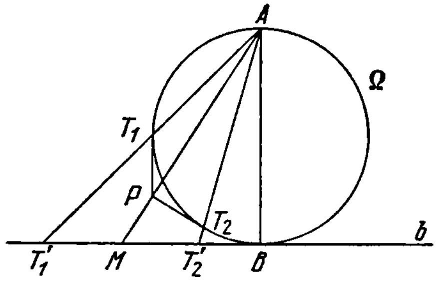

Fig. 104

Note that the projection M' of point P on the plane b_1 is not the image of point P under the inversion $I = [A, AB^2]$.

To construct the centre and the radius of the circle C', let us consider the section Ω of sphere Ω_1 by the plane ABP (Fig. 104). Let T_1', M and T_2' be the centre projections of points T_1, P, T_2 from point A on the straight line b (T_1 and T_2 are the points of tangency of tangents drawn from point P to the circle Ω). Then M is the centre of the circle C' and its radius is equal to $MT_1' = MT_2'$.

Let us now examine the case where the circle C is a great circle of Ω_1 and does not pass through point A. In this case, on the basis if the general properties of inversion, the stereographic projection C' of the circle C on the plane b_1 will again be a circle. Geometrically, its centre and radius are constructed as follows: the plane π in which circle C lies goes, under the inversion $I = [A, AB^2]$, into the sphere π', the centre of which lies on a straight line passing through point A perpendicular to plane π. On the other hand, since the plane π is orthogonal to the sphere Ω_1, it follows that the sphere π' must be orthogonal to the plane b_1 and, hence, the centre M of the sphere π' must lie in the plane b_1. Thus, the centre M of the sphere π' and, hence, of the circle C' as well (which is a great circle of the sphere π') is the point of intersection with the plane b_1 of a straight line passing through point A perpendicularly to the plane π in which the circle C lies.

Figure 105 shows a section Ω of the sphere Ω_1 cut by a plane passing through straight line AB and through the perpendicular dropped from point A to the plane π in which circle C lies; DE is a diameter of circle C along which the drawn plane intersects C. Joining point A with points D and E, we obtain, in the intersection with line b, the points D' and E', which are maps of the endpoints of diameter DE of circle C; the straight line AL, which is perpendicular to the diameter DE, intersects (as indicated

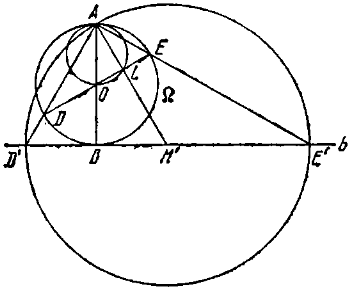

Fig. 105

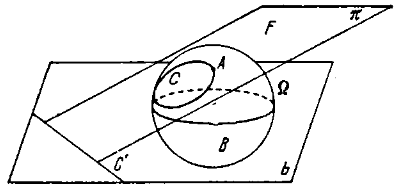

Fig. 106

above) the straight line b in the point M', which is the centre of circle C'. Since $M'D' = M'E' = M'A$, it follows that to construct the centre and the radius of circle C' we need not draw the straight lines AD and AE: by dropping the perpendicular AL from point A on the diameter DE of circle C we obtain both the centıe M' of circle C' and the radius AM' of that circle. Also note that the orthogonal projection L of point A on the straight line DE lies on the circle with diameter OA (O is the midpoint of line segment AB). This reasoning is used below in the geometric construction of a spectrum of meridians when constructing maps of the western and eastern hemispheres of the earth.

It remains to consider the case where the circle C passes through the centre A of projection. In this case, the stereographic projection of circle C is the straight line C' along which the plane b_1 intersects the plane π (in which circle C lies) (Fig. 106).

Let us now examine the construction of maps.

The geographic coordinates of a point lying on the earth's surface are called the latitude φ and the longitude θ. The lines on which the latitude φ is the same, $\varphi =$ constant, are termed parallels. These are sections of the earth by planes perpendicular to the axis NS of the poles. The lines on which the longitude θ has the same value, $\theta =$ constant, are termed the meridians of the earth. They are semicircumferences of the great circles of the earth, the boundary points of which are the poles N and S. The network of meridians and parallels of latitude on the earth's surface is an orthogonal network. Figure 107 depicts, in axonometric projection, the parallels of latitude and the meridians of a sphere.

Ordinarily, two methods are employed in the construction of a map of the earth with the use of stereographic projection: in one method the map of the northern and southern hemispheres is done as follows. Let N and S be the north pole and south pole, respectively, and let n and s be the planes tangent to the earth at these poles. To construct a stereographic projection of the northern hemisphere, one projects this hemisphere from the south pole S on the plane tangent to the earth at the north pole, and to construct a map of the southern hemisphere, one projects that

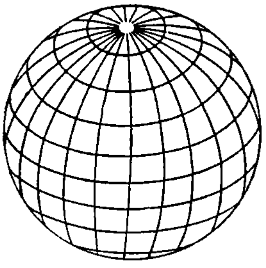

Fig. 107

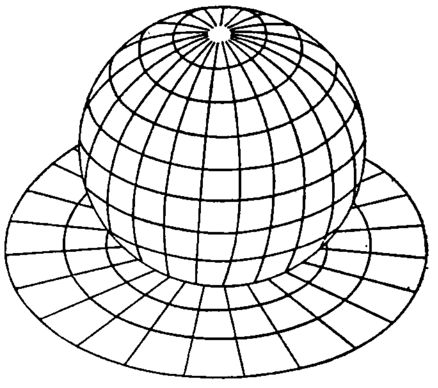

Fig. 108

hemisphere from point N on the plane s. The result is a map made up of two circles: one is a map of the northern hemisphere, the other is a map of the southern hemisphere. Of course, this construction is subjected to a similarity transformation. Figure 108 gives, in axonometric projection, just such a construction of the map of the southern hemisphere.

In such a mapping of the earth, the parallels of latitude of the southern hemisphere, for example, are projected into concentric circles lying in the plane s and having the common centre S; the equator σ_1 is projected into the circle σ inside which are the projections of all parallels. The semi-meridians of the southern hemisphere (or, similarly, of the northern hemisphere) are projected into radii of circles (on maps, this construction is of course supplemented by a similarity transformation). If we change the longitude θ at equal intervals (from 0° to 180°), then the corresponding radii of the circles σ will turn in succession through one and the same angle. Figure 109 depicts the construction of meridians under a change

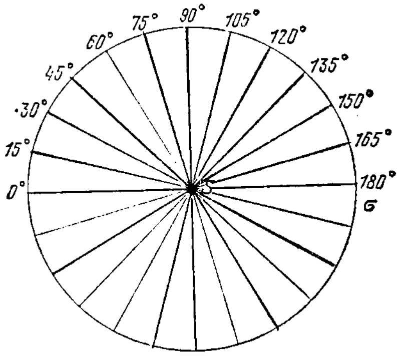

Fig. 109

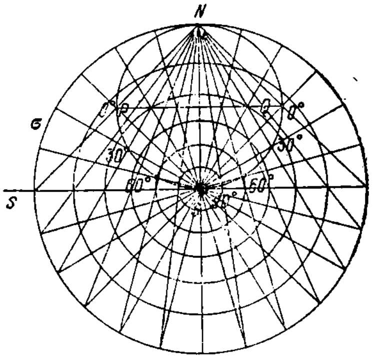

Fig. 110

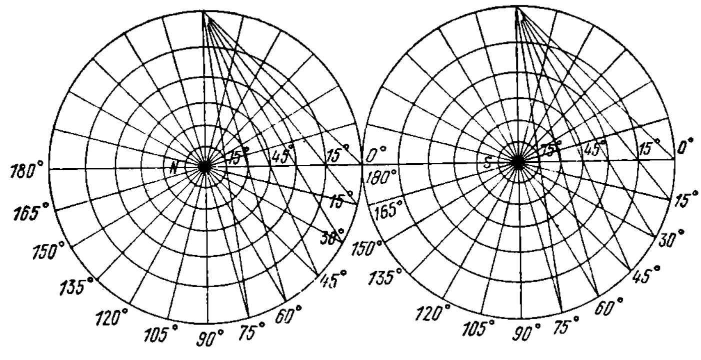

Fig. 111

in longitude θ of 15°. When constructing parallels corresponding to identical intervals of latitude (from 0° to 90°) one can do as follows: construct the section of the sphere (Fig. 110) by some plane passing through the straight line NS. Consider, for example, the southern hemisphere. Let PSQ be the semicircle of the southern hemisphere along which the drawn plane intersects the hemisphere, and let s be the straight line of the intersection of this plane with the plane s_1. Divide the semicircle PSQ into several equal parts (in Fig. 110 they are indicated by points with latitudes 0°, 15°, 30°, 45°, 60°, 75°, 90°); then the projections of the points of division from point N on straight line s with the same latitudes (0°0°, 15°15°, 30°30°, 45°45°, 60°60°, 75°75°, 90°90°) yield the diameters of the parallels. Construction of the parallels can be carried out together with that of the meridians. Just such a construction has been carried out in Fig. 110. Since the circle σ (the representation of the equator) is homothetic to circle (NS) with diameter NS (the homothetic ratio is equal to 2), it follows that when constructing meridians and parallels, one need not construct the circle (NS). That is the construction made in Fig. 111 for both hemispheres.

This geometric method of constructing meridians and parallels brings us to the conclusion that the distribution of meridians or, as we shall say, the spectrum of meridians, is uniform and the spectrum of parallels expands as we approach the representation σ of the equator (for the time being we note this property visually for parallels). We will give an analytic proof that the spectrum of parallels expands near the equator.

Let M be an arbitrary point of the sphere Ω, φ the latitude of point M, and M' the stereographic projection of point M from point N on the plane s_1. Let us now construct a section of the sphere Ω_1 by the plane NSM (Fig. 112). Since

$$\angle SOM = \frac{\pi}{2} - \varphi, \quad \text{it follows that } \angle SNM = \frac{\pi}{4} - \frac{\varphi}{2}$$

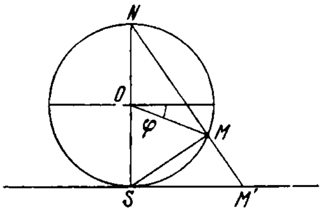

Fig. 112

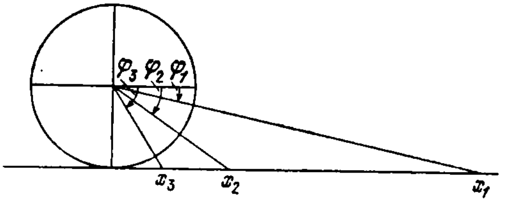

Fig. 113

and, consequently, (we assume that $NS = 1$)

$$x = SM' = \tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) = \cot\left(\frac{\pi}{4} + \frac{\varphi}{2}\right).$$

Now let us prove that if we take three points with latitudes $\varphi_1 < \varphi_2 < \varphi_3$ such that $\varphi_2 - \varphi_1 = \varphi_3 - \varphi_2$, then the corresponding values of the function x, that is,

$$x_1 = \cot\left(\frac{\pi}{4} + \frac{\varphi_1}{2}\right), \quad x_2 = \cot\left(\frac{\pi}{4} + \frac{\varphi_2}{2}\right), \quad x_3 = \cot\left(\frac{\pi}{4} + \frac{\varphi_3}{2}\right)$$

will be connected by the relation $x_1 - x_2 > x_2 - x_3$. This then is proof that the spectrum of representation of parallels is expanded near the representation of the equator (Fig. 113). The inequality $x_1 - x_2 > x_2 - x_3$, which we want to prove, is equivalent to $2x_2 < x_1 + x_3$ or

$$2\cot\left(\frac{\pi}{4} + \frac{\varphi_2}{2}\right) < \cot\left(\frac{\pi}{4} + \frac{\varphi_1}{2}\right) + \cot\left(\frac{\pi}{4} + \frac{\varphi_3}{2}\right)$$

or

$$2\,\frac{1 - \sin\varphi_2}{\cos\varphi_2} < \frac{\sin\left(\frac{\pi}{2} + \frac{\varphi_1 + \varphi_3}{2}\right)}{\sin\left(\frac{\pi}{4} + \frac{\varphi_1}{2}\right)\sin\left(\frac{\pi}{4} + \frac{\varphi_3}{2}\right)}$$

or

$$2\,\frac{1 - \sin\varphi_2}{\cos\varphi_2} < \frac{2\cos\varphi_2}{\cos\frac{\varphi_3 - \varphi_1}{2} - \cos\left(\frac{\pi}{2} + \frac{\varphi_1 + \varphi_3}{2}\right)}.$$

That is,

$$\frac{1 - \sin\varphi_2}{\cos\varphi_2} < \frac{\cos\varphi_2}{\cos\frac{\varphi_3 - \varphi_1}{2} + \sin\varphi_2}.$$

Since $\cos\varphi_2 > 0$, $\cos\frac{\varphi_3-\varphi_1}{2} > 0$, $\sin\varphi_2 > 0$, it follows that the inequality is equivalent to the following:

$$(1-\sin\varphi_2)\left(\cos\frac{\varphi_3-\varphi_1}{2}+\sin\varphi_2\right)<\cos^2\varphi_2$$

or

$$\cos\frac{\varphi_3-\varphi_1}{2}+\sin\varphi_2 < 1+\sin\varphi_2.$$

That is,

$$\cos\frac{\varphi_3-\varphi_1}{2} < 1.$$

This inequality is valid since $0^\circ < \frac{\varphi_3-\varphi_1}{2} < 90^\circ$; the inequality $x_1 - x_2 > x_2 - x_3$ is equivalent to it and, hence, it too is valid.

Remark. The very same result can be obtained by using derivatives. We have

$$x=\cot\left(\frac{\pi}{4}+\frac{\varphi}{2}\right),$$

$$x' = -\frac{1}{2}\,\frac{1}{\sin^2\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)} < 0.$$

Consequently, on the interval $[0, \pi/2]$, the function $x=\cot\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)$ is a decreasing function. Furthermore,

$$x'' = \frac{\cos\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}{2\sin^3\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)} > 0, \quad \varphi\in[0, \pi/2],$$

and so on the interval $[0, \pi/2]$ the graph of the function $x=\cot\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)$ is convex down and this means that on that interval the *Jensen inequality* holds true:

$$x_2 < \frac{x_1+x_2}{2}.$$

That is, $x_1 - x_2 > x_2 - x_3 > 0$ ($x_1 - x_2 > 0$, $x_2 - x_3 > 0$ since $0° \leqslant \leqslant \varphi_1 < \varphi_2 < \varphi_3 \leqslant 90°$, and on the interval $[0, \pi/2]$ the function $x = \cot\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$ is a decreasing function).

The established fact of the spectrum of parallels becoming sparser as one approaches the equator can also be substantiated approximately: the accompanying table contains the values of the function $x = \cot\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$ for the following values of φ: 0°, 15°, 30°, 45°, 60°, 75°, 90° and the values of $\Delta x = x_i - x_{i+1}$ ($i = 1, 2, 3, 4, 5, 6$). We see that the values of Δx diminish as the latitude increases, that is, as we recede from the equator

φ	x	Δx
0°	1,000	
		0.2327
15°	0.7673	
		0.1899
30°	0.5774	
		0.1632
45°	0.4142	
		0.1463
60°	0.2679	
		0.1362
75°	0.1317	
		0.1317
90°	0.0000	

Let us now examine the stereographic projections of the western and eastern hemispheres. Let α be the meridian from which we reckon the longitudes θ, and WV the diameter of the earth perpendicular to the plane in which the meridian α lies. The stereographic projection of the western hemisphere (Fig. 114 shows the lower half of the sphere) is obtained by projecting the western hemisphere from point W on the plane β, which is tangent to the western hemisphere at the point V. Since such projection coincides with the inversion $I = [W, WV^2]$ for points of the sphere Ω_1, it follows that the meridians and the parallels, as forming an orthogonal network on the earth, will project into an orthogonal network of circles on the plane β. Here, the meridians of each of the hemispheres (in Fig. 114 we have the western hemisphere) will project into arcs of circles passing through the points N and S for which these points serve as boundary points. Note that the spectrum of representations of the meridians (of any one of the hemispheres) is exactly the same as the spectrum of parallels

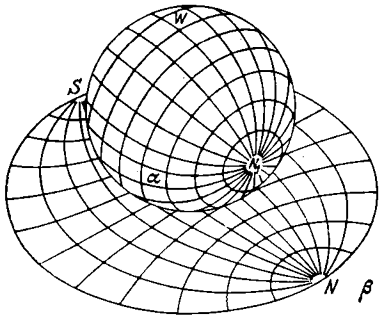

Fig. 114

under the foregoing stereographic projection of parallels of the northern and southern hemispheres. However, the centres of the arcs representing the meridians of each of the hemispheres must be constructed in accord with Fig. 115 which shows that construction (a map of the western and eastern hemispheres). As for parallels, their construction should be based on the following reasoning: since meridians and parallels form an orthogonal network, this network must also remain orthogonal in the representation of the sphere under a stereographic projection. In particular, the parallels must be orthogonal to the representation of their principal meridian α. Besides, the centres of parallels lie on the axis NS of the N and S poles. From this we obtain the following method of a geometric construction of a parallel passing through a given point M of the principal meridian α: at point M draw a tangent to the circle α (to the representation of the principal meridian). Suppose P is the point in which this tangent line intersects the straight line NS; then the circle with centre at point P and radius PM will be the representation of the parallel passing through point M (Fig. 116). In Fig. 117 are constructed the representations of the parallels (and meridians, in accord with Fig. 115, that correspond to the values of latitude 0°, 15°, 30°, 45°, 60°, 75° for both hemispheres — western and eastern).

From Fig. 117 it is evident that the spectrum of parallels expands near the poles. We will now prove this analytically. Let φ be the latitude of point M (Fig. 116). Setting $NS = 2$, we have $PM = \cot\varphi$, $OP = \operatorname{cosec}\varphi$ and, hence,

$$y = OK = OP - PM = \operatorname{cosec}\varphi - \cot\varphi = \frac{1-\cos\varphi}{\sin\varphi} = \tan\frac{\varphi}{2}\cdot$$

This function $y = \tan(\varphi/2)$ is an increasing function on the interval $[0, \pi/2]$ and as φ varies from 0 to $\pi/2$, the function $y = \tan(\varphi/2)$ increases from 0 to 1. We will prove that the values of the latitudes $\varphi_1, \varphi_2, \varphi_3$, which have equal increments (on the interval $0 \leqslant \varphi_1 < \varphi_2 < \varphi_3 \leqslant \pi/2$; $\varphi_2 -$

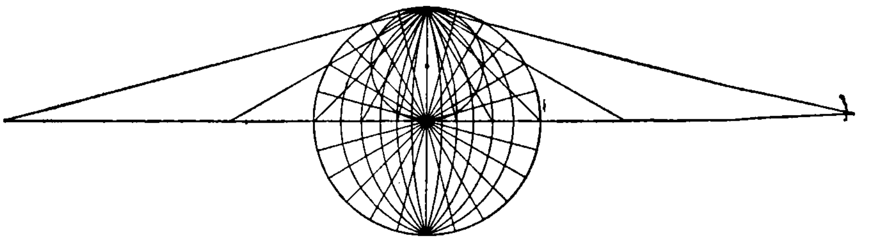

Fig. 115

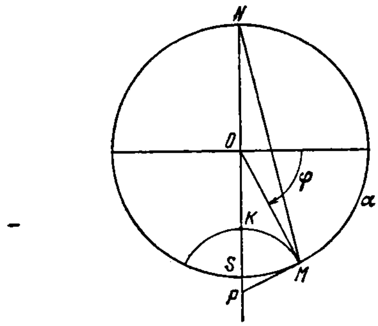

Fig. 116

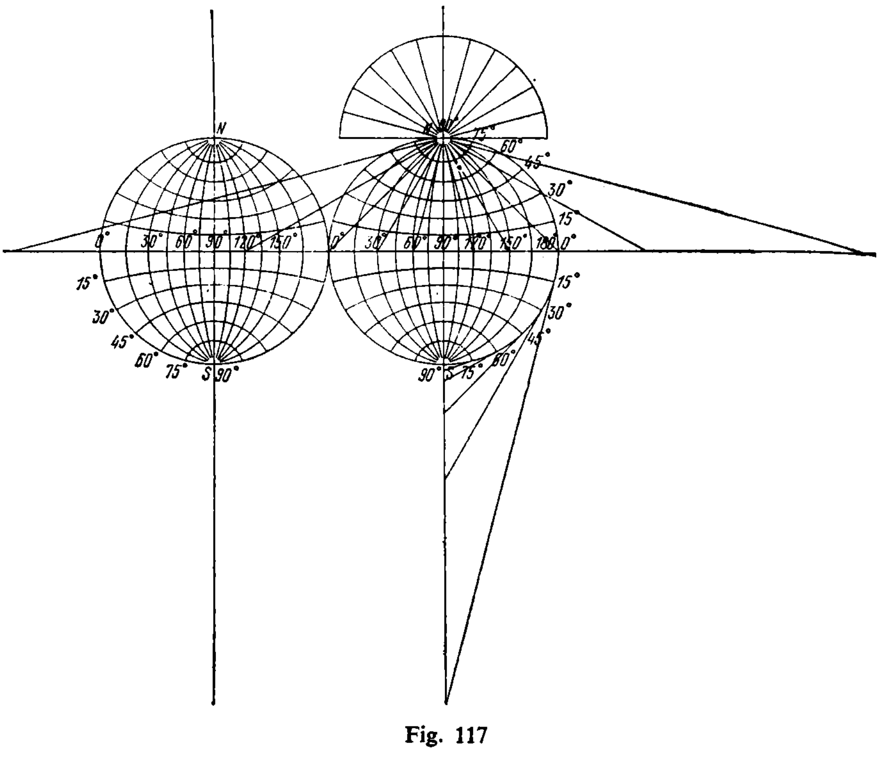

Fig. 117

$-\varphi_1 = \varphi_3 - \varphi_2 > 0$), that is, $\varphi_2 = (\varphi_1 + \varphi_3)/2$, are associated with the values of the function

$$y_1 = \tan(\varphi_1/2),\ y_2 = \tan(\varphi_2/2),\ y_3 = \tan(\varphi_3/2)$$

such that $y_2 - y_1 < y_3 - y_2$ or that $2y_2 < y_1 + y_3$. Such is the proof that the spectrum of parallels becomes less dense near the poles. Thus, we have to prove that

$$2\tan(\varphi_2/2) < \tan(\varphi_1/2) + \tan(\varphi_2/2),$$

that is,

$$2\,\frac{\sin(\varphi_2/2)}{\cos(\varphi_2/2)} < \frac{\sin((\varphi_1 + \varphi_3)/2)}{\cos(\varphi_1/2)\cos(\varphi_3/2)}$$

or

$$2\,\frac{\sin(\varphi_2/2)}{\cos(\varphi_2/2)} < \frac{\sin\ \varphi_2}{\cos(\varphi_1/2)\cos(\varphi_3/2)},$$

which is equivalent to

$$\cos(\varphi_1/2)\cos(\varphi_3/2) < \cos^2(\varphi_2/2)$$

or

$$\cos\varphi_2 + \cos\frac{\varphi_3 - \varphi_1}{2} < 1 + \cos\varphi_2,$$

that is,

$$\cos\frac{\varphi_3 - \varphi_1}{2} < 1.$$

This inequality is valid since $0 < (\varphi_3 - \varphi_1)/2 < \pi/2$; the inequality $y_2 < (y_1 + y_3)/2$ is equivalent to it and, hence, is true as well.

The same values can be obtained if one uses derivatives. We have

$$y = \tan(\varphi/2),$$

$$y' = \frac{1}{2\cos^2(\varphi/2)} > 0, \qquad \varphi \in [0, \pi/2],$$

and so the function $y = \tan(\varphi/2)$ increases on the interval $[0, \pi/2]$. Furthermore,

$$y'' = \frac{\sin(\varphi/2)}{2\cos^3(\varphi/2)} > 0, \qquad \varphi \in [0, \pi/2],$$

consequently, the graph of the function $y = \tan(\varphi/2)$ on the interval $[0, \pi/2]$ is convex down, and therefore on that interval the following Jensen inequality holds true: $y_2 < (y_1 + y_3)/2$, where y_1, y_2, y_3 have the above-indicated values.

Finally, the fact that the spectrum of parallels expands near the poles is confirmed by the following table too, in which are given the values of the function $y = \tan(\varphi/2)$ at 15° intervals of latitude and the values of $\Delta y = y_{i+1} - y_i$ $(i = 1, 2, 3, 4, 5, 6)$.

φ	y	Δy
0°	0.0000	
		0.1317
15°	0.1317	
		0.1362
30°	0.2679	
		0.1463
45°	0.4142	
		0.1632
60°	0.5774	
		0.1899
75°	0.7673	
		0.2327
90°	1.0000	

Incidentally, all of this immediately follows from the fact that the spectrum of the function $y = \tan(\varphi/2)$ repeats "in inverse order", as φ varies from 0° to 90°, the spectrum of the function

$$x = \cot\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) = \tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)$$

$\Big[$compare the tables of the values of the functions $y = \tan\dfrac{\varphi}{2}$ and $x = \tan\left(\dfrac{\pi}{4} - \dfrac{\varphi}{2}\right)\Big]$.

If we complete the representations of the meridians and parallels to full circles when constructing maps of the western and eastern hemispheres (in stereographic projection), then we obtain a set of all circles passing through the points N and S (meridians) and a set of all circles orthogonal to the representations of the meridians (parallels) (Fig. 118).

The set of all circles passing through two fixed points N and S is called an *elliptic pencil of circles;* the points N and S are then termed *base* points. The set of all circles, each of which is orthogonal to all circles of the elliptic pencil with base points N and S is termed a *hyperbolic pencil of circles,* which is the conjugate of the elliptic pencil; the points N and S are termed *limit* points of the hyperbolic pencil, or *Poncelet points* of such a pencil. When representing a network of meridians and parallels under a stereographic projection of the western and eastern hemispheres, we have to

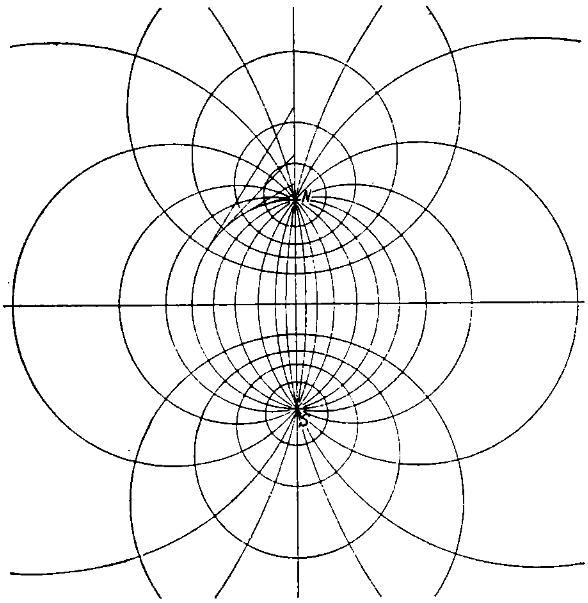

Fig. 118

isolate from the elliptic pencil of circles and the conjugate hyperbolic pencil that part contained within the circle of the elliptic pencil, whose diameter is the line segment NS, where N and S are base points of the elliptic pencil.

A stereographic projection of a network of meridians and parallels may be carried out by chosing, on a sphere, any two diametrically opposed points A and B and projecting the sphere from point A on a plane tangent to the sphere Ω_1 at the point B. Since a stereographic projection is a conformal mapping of the sphere on a plane, it follows that the orthogonal network of meridians and parallels will be mapped into the same kind of orthogonal network of two pencils of circles: the meridians are mapped into an elliptic pencil with base points N and S, and the parallels into the conjugate hyperbolic pencil with limit points N and S. The projection of the principal meridian is some circle K that passes through the points N and S, and NS is a chord of that circle. A map of one of the hemispheres (western or eastern) will lie inside circle K. The construction of the map of the other hemisphere is similar.

If we isolate any one of the circles (say circle α) from the hyperbolic pencil of circles and regard it as a representation of the equator (and regard the Poncelet point lying inside the chosen circle α as a representation of the pole), then we obtain a map of the northern or southern hemisphere. Representations of the parallels are all the cilcles of the hyperbolic pencil that lie inside the circle α, and the semimeridians are arcs of

the conjugate elliptic pencil that lie inside the circle α and emanate from the pole (located inside α). In this situation, these semimeridians will join up into an arc of one circle if the sum of the longitudes (east and west) is equal to 180° (or, to put it differently, if the difference between the positive and negative longitude is equal to 180°).

Such is the qualitative picture of representation of meridians and parallels under an arbitrary stereographic projection of the western, eastern, northern and southern hemispheres.

We will now show how an exact geometric construction is performed of the spectrum of meridians and parallels under an arbitrary stereographic projection.

Let us begin with the construction of the spectrum of meridians and parallels when representing the western (or eastern) hemisphere; we assume the latitude and longitude to vary in 15° intervals. Consider the section $\Omega = ANBS$ of the sphere Ω_1 by a plane passing through the diameters AB and SN (Fig. 119). Let N and S be projections of points N and S from point A on a tangent to the section Ω at the point B. Since the meridians pass one into another (and the parallels are invariant) under rotation of the sphere Ω_1 about the axis SN, it follows that any circle with chord NS may be taken for the representation α of the principal meridian with a supplement to complete the meridian to a full circle. Since a stereographic projection is a conformal mapping of the sphere on a plane, in order to construct the representation of the meridians in equal intervals of longitude, one has to construct arcs of circles whose centres lie on the midperpendicular of line segment NS, the centres lying inside the representation α of the principal meridian and forming a sequence of equal angles. For example, if the longitude changes in 15° intervals, then we obtain the representations of thirteen meridians (the circle α is a representation of two meridians with longitudes of 0° and 180°), which intersect in succession at 15° angles. In Fig. 120, the constructions have been carried out in superposed planes: the plane of the circle Ω is brought to coincidence with the plane tangent to the sphere Ω_1 at the point B (the coincidence is attained by rotating about the straight line along which these two planes intersect).

For a geometric construction of the representation of these meridians, note that the angle between two intersecting circles is equal to the angle between their radii drawn to the point of intersection of the circles. For this reason, to construct the radii and centres of the representations of the meridians, turn the radius NO of the circle α about point N in a succession of, say, 15° angles. To do this, construct the semicircle CDE with centre N, whose diameter CE is tangent to the circle α at the point N (see Fig. 119). Divide the arc CDE into 12 equal parts and project the points of division from point N onto the midperpendicular of line segment NS. These projections T are the centres of the representations of the meridians (which correspond to 15° intervals of longitude), and TN are their radii.

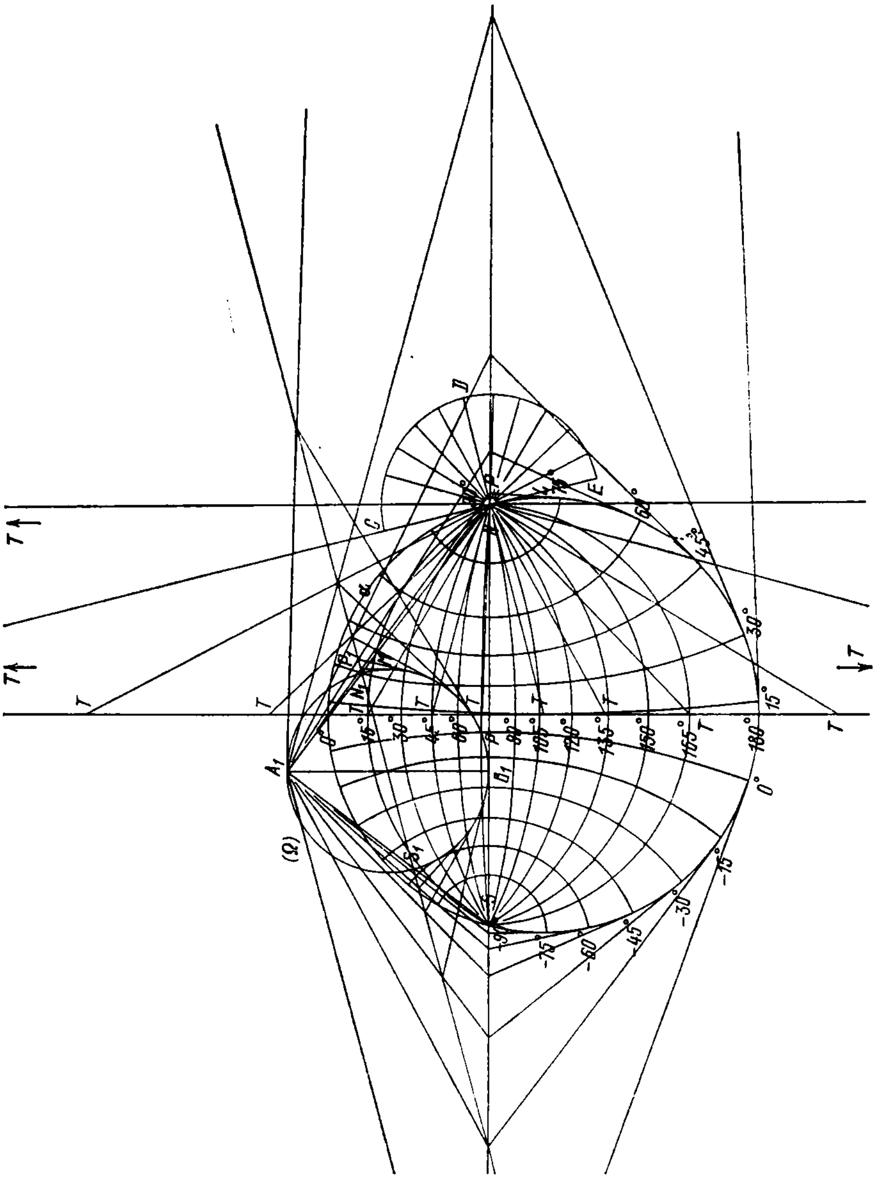

Fig. 119

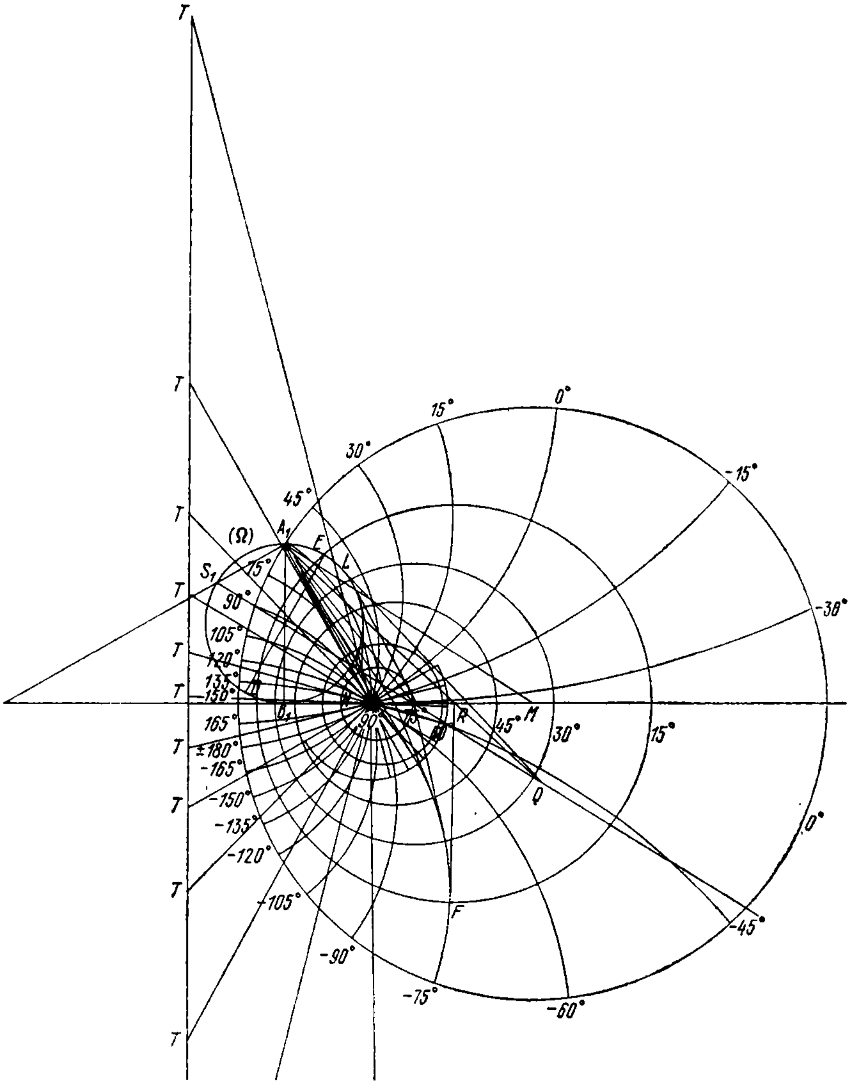

Fig. 120

Note that the idea of conformity of a stereographic projection that was made use of in this construction could have been utilized earlier in the construction of the spectrum of meridians in a simpler case.

To construct the representations of parallels at, say, 15° intervals of latitude, one should take advantage of the method of constructing the

centre of a stereographic projection of a circle that was given above for an arbitrary diameter AB of the sphere Ω_1 with respect to the circle lying on that sphere. Let OT be a radius of the circle Ω perpendicular to S_1N_1. Divide the arc S_1TN_1 into 12 equal parts. Let M be one of the points of division (say, the closest one to N_1). The tangent to the circle Ω at point M intersects the straight line S_1N_1 in the vertex P of a cone tangent to the sphere Ω_1 along a parallel passing through point M. The projection P' of point P from point A onto the straight line NS will be the centre of the representation of that parallel. If PL is the tangent, for example, to the representation α of the principal meridian (L is the point of tangency), then PL is a radius of the representation of the parallel at hand (and P is its centre). Thus, Fig. 120 depicts the representations of half the parallels and half the equator (the centre of the representation of the equator is the point of intersection of the straight line NS with the straight line passing through point A parallel to N_1S_1).

Let us now consider the construction of the spectrum of meridians and parallels under a stereographic projection, say, of the northern hemisphere for an arbitrary position of the diameter AB of the sphere Ω_1 with respect to the equator.

Again construct the section Ω of sphere Ω_1 by the plane AN_1BS_1 and let DE be a diameter of the circle Ω along which the plane of the section cuts the plane of the equator. Let N and S be the projections of the points N_1 and S_1 from point A on the tangent line to the circle Ω at the point B. To make a representation of the equator, drop from point A a perpendicular to the diameter DE. Let M be the point of intersection of this perpendicular with the tangent to the circle Ω at the point B. Then M is the centre of the representation of the equator, and MA is the radius of that representation.

We will perform all subsequent constructions in the superposed planes: the plane of the circle Ω with the plane tangent to the sphere Ω_1 at point B (as above, coincidence is attained by turning about the straight line along which these two planes intersect).

To construct the representations of parallels at, say, 15° intervals of latitude, divide the arc EN_1 of the circle Ω into 6 equal parts. Let L be one of the points of division (for example, the one closest to E). If Q is the point of intersection of the tangent to the circle Ω at point L with the straight line S_1N_1, then Q is the vertex of a cone that is tangent to the sphere Ω_1 along the parallel passing through point L. The projection R of point Q on the straight line BN from point A is the centre of the representation of the parallel that passes through point L. Now the radius of the representation of this parallel is equal to the line segment RF of the tangent drawn to the representation of any meridian (that is, to any circle passing through the points N and S). The foregoing method is used to construct the parallels of the northern hemisphere for 15-degree intervals of latitudes from 90° to 0°.

To construct the representations of meridians, note (we have already mentioned this) that when the sphere is rotated about the axis N_1S_1, the meridians pass into one another (while the parallels are invariant). Therefore, in order to construct semimeridians we can take, for the two principal semimeridians that correspond to longitudes of 0° and 180° and form a single arc of the circle, any arc of the circle that passes through the points N and S and lies inside the representation of the equator.

Construct a semicircle with centre N, divide it, say, into 12 equal parts and project the points of division from point N onto the midperpendicular of the line segment SN (exactly the same approach was used above to construct the representations of the meridians under a stereographic projection of the western and eastern hemispheres). The projections will be the centres of semimeridians emanating from point N and forming a succession of 15-degree angles. The semimeridians, the difference of the longitudes of which is equal to 180° ($180° - 0° = 180°, 165° - (-15°) = 180°, 150° - (-30°) = 180°$ and so on) are joined in the representation into one single arc of the circle, which arc lies inside the representation of the equator. This occurs, for instance, in the most elementary case as well (it is shown in the figure), where the representations of the meridians (that is, the radii of a circle) for the values of the longitudes (the difference of which is equal to 180°) are joined into a diameter of the circle, which is the representation of the equator. Of course, the same also occurs on the sphere Ω_1 on any one of the two hemispheres (northern or southern).

CHAPTER V

BASIC DEFINITIONS, THEOREMS AND FORMULAS

Sec. 1. Determinants of order three

A *third-order determinant* is introduced by the following relation:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 + a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2; \quad (1)$$

a_i, b_i, c_i $(i = 1, 2, 3)$ are termed the *elements* of the determinant.

Basic properties:

1°. A determinant remains unchanged under a *transposition*, that is, under an interchange of rows and columns:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \quad (2)$$

2°. A determinant preserves its absolute value but changes sign when any two columns are interchanged; for example:

$$\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \quad (3)$$

A determinant remains unchanged under a cyclic permutation of its columns:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix} = \begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix} \quad (4)$$

and changes sign if the cyclic order of its columns is disrupted [see (3)].

3°. If two columns of a determinant are the same, the determinant is equal to zero.

4°. If all the elements of some column of a determinant are equal to zero, the determinant is zero.

5°. A determinant is a linear function of its columns. This means that the following relation holds true:

$$\begin{vmatrix} \lambda a_1 + \mu k_1 & b_1 & c_1 \\ \lambda a_2 + \mu k_2 & b_2 & c_2 \\ \lambda a_3 + \mu k_3 & b_3 & c_3 \end{vmatrix} = \lambda \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \mu \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}, \tag{5}$$

and analogous relations for the second and third columns.

6°. A determinant remains unchanged if a linear combination of any two columns is added to another column, that is,

$$\begin{vmatrix} a_1 + \lambda b_1 + \mu c_1 & b_1 & c_1 \\ a_2 + \lambda b_2 + \mu c_2 & b_2 & c_2 \\ a_3 + \lambda b_3 + \mu c_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \tag{6}$$

and analogously for the second and third columns. In particular, a determinant remains unchanged if any column is added to or subtracted from another column.

7°. For a determinant to be equal to zero, it is necessary and sufficient that its columns be linearly dependent, that is, that there exist numbers λ, μ, ν among which at least one is nonzero and such that

$$\begin{aligned} \lambda a_1 + \mu b_1 + \nu c_1 &= 0, \\ \lambda a_2 + \mu b_2 + \nu c_2 &= 0, \\ \lambda a_3 + \mu b_3 + \nu c_3 &= 0. \end{aligned} \tag{7}$$

In other words, for a determinant to be equal to zero, it is necessary and sufficient that one of the columns be a linear combination of the other two, for example:

$$\begin{aligned} c_1 &= \lambda a_1 + \mu b_1, \\ c_2 &= \lambda a_2 + \mu b_2, \\ c_3 &= \lambda a_3 + \mu b_3. \end{aligned} \tag{8}$$

8°. Multiplication of determinants is carried out with the aid of the following formula:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 x_1 + b_1 x_2 + c_1 x_3 & a_1 y_1 + b_1 y_2 + c_1 y_3 & a_1 z_1 + b_1 z_2 + c_1 z_3 \\ a_2 x_1 + b_2 x_2 + c_2 x_3 & a_2 y_1 + b_2 y_2 + c_2 y_3 & a_2 z_1 + b_2 z_2 + c_2 z_3 \\ a_3 x_1 + b_3 x_2 + c_3 x_3 & a_3 y_1 + b_3 y_2 + c_3 y_3 & a_3 z_1 + b_3 z_2 + c_3 z_3 \end{vmatrix}. \tag{9}$$

Since a determinant remains unchanged under the transpose operation, we can obtain another three modes of multiplication of determinants if one of the factors of the left-hand side of (9) is transposed.

The *cofactor* of an element of a determinant is the coefficient of tha element in formula (1).

For example, the cofactor C_1 of element c_1 in determinant (1) is

$$C_1 = a_2b_3 - a_3b_2.$$

The cofactor A_2 of element a_2 of the determinant (1) is

$$A_2 = b_3c_1 - b_1c_3$$

and so forth [in all, there are 9 cofactors A_i, B_i, C_i $(i = 1, 2, 3)$].

A determinant is equal to the sum of the products of the elements of any column into the corresponding cofactors:

$$\Delta = a_1A_1 + a_2A_2 + a_3A_3,$$

and similarly for the second and third columns.

A determinant is equal to the sum of the products of the elements of any row into the corresponding cofactors:

$$\Delta = a_1A_1 + b_1B_1 + c_1C_1,$$

and similarly for the second and third rows.

The sum of the products of the elements of any column into the corresponding cofactors of the elements of another column is equal to zerot, for example:

$$a_1B_1 + a_2B_2 + a_3B_3 = 0$$

(and there are five other analogous relations).

The sum of the products of the elements of any row into the corresponding cofactors of the elements of another row is equal to zero, for example:

$$a_1A_2 + b_1B_2 + c_1C_2 = 0$$

(and there are five other analogous relations).

The *minor* of an element of a determinant is the determinant obtained by deleting the row and column at the intersection of which that element lies.

For example, the minor of element c_1 in determinant (1) is the determinant

$$\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = a_2b_3 - a_3b_2$$

and the minor of element a_2 of determinant (1) is the determinant

$$\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} = b_1c_3 - b_3c_1, \text{ and so on.}$$

The cofactor A of any element of a determinant is equal to the minor M of that element multiplied by $(-1)^{i+j}$, where i and j are, respectively,

the number of the row and the number of the column, at the intersection of which that element lies:

$$A = (-1)^{i+j} M.$$

Sec. 2. Vector algebra

A *vector* is a directed line segment. Vectors are symbolized as follows: $\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{r}, \ldots$ or $\overrightarrow{AB}, \overrightarrow{CD}, \overrightarrow{PQ}, \ldots$

A vector $\overrightarrow{AB}$ is said to be *nonzero* if the points A and B are distinct; a vector $\overrightarrow{AB}$ is said to be a *zero* vector if the points A and B coincide. The zero vector is designated by the symbol $\mathbf{0}$.

Two nonzero vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to be *collinear* if the straight lines AB and CD are collinear, that is, either parallel or coincident. The zero vector is assumed to be collinear with any other vector.

If the vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear, then we can write $\mathbf{a} \parallel \mathbf{b}$. If the vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear and in the same direction, we can write $\mathbf{a} \uparrow\uparrow \mathbf{b}$, and if they have opposite directions we write $\mathbf{a} \uparrow\downarrow \mathbf{b}$.

The *magnitude* (or *absolute value)* of a vector $\overrightarrow{AB}$ is the length of the line segment AB. This can be symbolized as: $|\overrightarrow{AB}|$, AB, $|\mathbf{a}|$, or a.

The nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be *equal* if $\mathbf{a} \uparrow\uparrow \mathbf{b}$ and $|\mathbf{a}| = |\mathbf{b}|$. The zero vector is assumed to be different from any nonzero vector. Zero vectors are all assumed to be equal.

The nonzero vectors $\overrightarrow{AB}, \overrightarrow{CD}, \overrightarrow{EF}$ are said to be *coplanar* if the straight lines AB, CD and EF are coplanar to one and the same plane (a *straight line* and a *plane* are said to be *coplanar* if the line is either parallel to, or lies in, the plane). If there is at least one zero vector among the vectors $\overrightarrow{AB}, \overrightarrow{CD}, \overrightarrow{EF}$, then they are regarded as being coplanar.

To lay off a vector $\mathbf{a}$ from a given point A means to construct a directed line segment $\overrightarrow{AB}$ equal to the vector $\mathbf{a}$.

The vector $-\mathbf{a} = \overrightarrow{BA}$ is said to be the opposite of vector $\mathbf{a} = \overrightarrow{AB}$.

The *sum* $\mathbf{a} + \mathbf{b}$ of the vectors $\mathbf{a}$ and $\mathbf{b}$ is a vector which is constructed as follows:
from an arbitrary point A lay off the vector $\mathbf{a}$:

$$\overrightarrow{AB} = \mathbf{a}$$

and from the point B lay off the vector $\mathbf{b}$:

$$\overrightarrow{BC} = \mathbf{b}.$$

Then

$$\mathbf{a} + \mathbf{b} = \overrightarrow{AC}.$$

The properties of a sum of vectors are:

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c},$$

$$\mathbf{a} + \mathbf{0} = \mathbf{a},$$

$$\mathbf{a} + (-\mathbf{a}) = \mathbf{0},$$

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}.$$

The *difference* $\mathbf{a} - \mathbf{b}$ between two vectors is a vector $\mathbf{x}$ such that $\mathbf{b} + \mathbf{x} = \mathbf{a}$. To construct the difference $\mathbf{a} - \mathbf{b}$, lay off vectors $\mathbf{a}$ and $\mathbf{b}$ from some point O:

$$\overrightarrow{OA} = \mathbf{a}, \quad \overrightarrow{OB} = \mathbf{b}.$$

Then $\mathbf{a} - \mathbf{b} = \overrightarrow{BA}$.

The *product* $\lambda\mathbf{a}$ *of a number* $\lambda \neq 0$ *by a vector* $\mathbf{a} \neq \mathbf{0}$ is a vector defined as follows: $|\lambda\mathbf{a}| = |\lambda|\,|\mathbf{a}|$, and we have $\lambda\mathbf{a} \uparrow\uparrow \mathbf{a}$ if $\lambda > 0$, and $\lambda\mathbf{a} \downarrow\uparrow \mathbf{a}$ if $\lambda < 0$. If either $\lambda = 0$ or $\mathbf{a} = \mathbf{0}$, then, by definition, $\lambda\mathbf{a} = \mathbf{0}$.

The properties of a product of a number (scalar) by a vector are:

$$1 \cdot \mathbf{a} = \mathbf{a},$$

$$\lambda(\mu\mathbf{a}) = (\lambda\mu)\,\mathbf{a},$$

$$\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b},$$

$$(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}.$$

If $\mathbf{a} \parallel \mathbf{b} \neq \mathbf{0}$, then the *ratio* $\dfrac{\mathbf{a}}{\mathbf{b}}$ is a number λ such that $\lambda\mathbf{b} = \mathbf{a}$. If $\mathbf{0} \neq \mathbf{a} \parallel \mathbf{b} \neq \mathbf{0}$, then $\dfrac{\mathbf{a}}{\mathbf{b}} = \pm \dfrac{|\mathbf{a}|}{|\mathbf{b}|}$, the sign being "+" if $\mathbf{a} \uparrow\uparrow \mathbf{b}$ and the sign being "−" if $\mathbf{a} \uparrow\downarrow \mathbf{b}$. If $\mathbf{a} = \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$, then $\dfrac{\mathbf{a}}{\mathbf{b}} = 0$.

A *linear combination* of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ is the sum $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c} + \ldots$

The vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ are said to be *linearly dependent* if there are numbers λ, μ, ν such that at least one of them is different from zero and such that $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c} + \ldots = \mathbf{0}$. If this equation is possible only for $\lambda = \mu = \nu = \ldots = 0$, then the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are said to be *linearly independent*.

For the vectors $\mathbf{a}$ and $\mathbf{b}$ to be collinear, it is necessary and sufficient that they be linearly dependent.

For the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ to be coplanar, it is necessary and sufficient that they be linearly dependent.

A *basis* $\mathbf{e}_1$, $\mathbf{e}_2$ in a plane is an ordered pair of noncollinear vectors $\mathbf{e}_1$, $\mathbf{e}_2$ coplanar to that plane. If the vectors of the basis are unit vectors and are mutually perpendicular, the basis is said to be *orthonormal*. Ordinarily, the vectors of an orthonormal basis are designated as $\mathbf{i}$, $\mathbf{j}$:

$$\mathbf{i} \perp \mathbf{j}, \quad |\mathbf{i}| = |\mathbf{j}| = 1.$$

Suppose a basis $\mathbf{e}_1$, $\mathbf{e}_2$ has been introduced in a plane; $\mathbf{a}$ is an arbitrary vector coplanar to that plane. There exists a pair of numbers x, y (and only one pair) such that

$$\mathbf{a} = x\mathbf{e}_1 + y\mathbf{e}_2.$$

The coefficients x and y of $\mathbf{e}_1$ and $\mathbf{e}_2$ in this expansion of the vector $\mathbf{a}$ with respect to the basis $\mathbf{e}_1$, $\mathbf{e}_2$ are called the *components* (or *coordinates*) of vector $\mathbf{a}$ in (or with respect to) the basis $\mathbf{e}_1$, $\mathbf{e}_2$. If the vector $\mathbf{a}$ in the basis $\mathbf{e}_1$, $\mathbf{e}_2$ has components x, y, then we write $\mathbf{a} = \{x, y\}$ or $\mathbf{a}\ \{x, y\}$.

In space, the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is formed by an ordered triplet of noncoplanar vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.

If the vectors of a basis are unit vectors and orthogonal in pairs, then the basis is said to be *orthonormal*. The vectors of an orthonormal basis are ordinarily designated as $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$:

$$\mathbf{i} \perp \mathbf{j}, \quad \mathbf{j} \perp \mathbf{k}, \quad \mathbf{k} \perp \mathbf{i}, \quad |\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1.$$

If we introduce a basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ in space and let $\mathbf{a}$ be an arbitrary vector, then there is only one triplet of numbers x, y, z such that

$$\mathbf{a} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3.$$

The numbers x, y, z are called the *components* (or *coordinates*) of the vector $\mathbf{a}$ in the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. If the vector $\mathbf{a}$ has components x, y, z, then we write $\mathbf{a} = \{x, y, z\}$ or $\mathbf{a}\ \{x, y, z\}$.

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are *equal* if and only if their corresponding components are equal.

If, in the basis $\mathbf{e}_1$, $\mathbf{e}_2$, we have

$$\mathbf{a} = \{x, y\}, \qquad \mathbf{b} = \{x', y'\},$$

then

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \{x + x',\ y + y'\}, \\ \mathbf{a} - \mathbf{b} &= \{x - x',\ y - y'\}, \\ \lambda\mathbf{a} &= \{\lambda x, \lambda y\}. \end{aligned}$$

If, in the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, we have

$$\mathbf{a} = \{x,\ y,\ z\}, \qquad \mathbf{b} = \{x',\ y',\ z'\},$$

then

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \{x + x',\ y + y',\ z + z'\}, \\ \mathbf{a} - \mathbf{b} &= \{x - x',\ y - y',\ z - z'\}, \\ \lambda\mathbf{a} &= \{\lambda x, \lambda y, \lambda z\}. \end{aligned}$$

A necessary and sufficient condition for two vectors $\mathbf{a} = \{x, y\}$ and $\mathbf{b} = \{x', y'\}$ to be collinear in a plane is the equation

$$\begin{vmatrix} x & x \\ x' & y' \end{vmatrix} = 0$$

A necessary and sufficient condition for two vectors $\mathbf{a} = \{x, y, z\}$ and $\mathbf{b} = \{x' y', z'\}$ to be collinear in space is the condition

$$\begin{vmatrix} y & z \\ y' & z' \end{vmatrix} = \begin{vmatrix} z & x \\ z' & x' \end{vmatrix} = \begin{vmatrix} x & y \\ x' & y' \end{vmatrix} = 0.$$

A necessary and sufficient condition for the coplanarity of three vectors

$$\mathbf{a} = \{x, y, z\}, \quad \mathbf{b} = \{x', y', z'\}, \quad \mathbf{c} = \{x'', y'', z''\}$$

is the equation

$$\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = 0.$$

A *scalar product* $\mathbf{a} \cdot \mathbf{b}$ (or $\mathbf{ab}$) of two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ is the product of their magnitudes by the cosine of the angle between them:

$$\mathbf{ab} = |\mathbf{a}|\,|\mathbf{b}| \cos \varphi.$$

If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, then, by definition, $\mathbf{ab} = 0$.

The properties of a scalar product of two vectors are:

$$\mathbf{a}^2 = \mathbf{aa} > 0 \quad \text{if} \quad \mathbf{a} \neq \mathbf{0}, \quad \mathbf{a}^2 = 0 \quad \text{if} \quad \mathbf{a} = \mathbf{0},$$

$$\mathbf{ab} = \mathbf{ba},$$

$$(\lambda \mathbf{a})\,\mathbf{b} = \lambda(\mathbf{ab}),$$

$$\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{ab} + \mathbf{ac}.$$

The scalar product $\mathbf{aa}$ (ordinarily we write $\mathbf{a}^2$) is called the scalar square of the vector $\mathbf{a}$.

If, in an orthonormal basis $\mathbf{i}$, $\mathbf{j}$ in the plane,

$$\mathbf{a} = \{x, y\}, \qquad \mathbf{b} = \{x', y'\},$$

then

$$\mathbf{ab} = xx' + yy',$$

and if

$$\mathbf{a} = \{x, y\}, \qquad \mathbf{b} = \{x', y'\}$$

in an arbitrary basis $\mathbf{e}_1$, $\mathbf{e}_2$, then

$$\mathbf{ab} = g_{11}xx' + g_{12}(xy' + x'y) + g_{22}\,yy',$$

where

$$g_{ij} = \mathbf{e}_i\,\mathbf{e}_j.$$

The collection of scalar products $g_{ij} = \mathbf{e}_i\,\mathbf{e}_j$ of the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$ is termed the *fundamental tensor* of that basis.

If, in the orthonormal basis $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$,

$$\mathbf{a} = \{x,\ y,\ z\}, \qquad \mathbf{b} = \{x',\ y',\ z'\},$$

then

$$\mathbf{ab} = xx' + yy' + zz',$$

and if

$$\mathbf{a} = \{x, y, z\}, \qquad \mathbf{b} = \{x', y', z'\}$$

in the arbitrary basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, then

$$\mathbf{ab} = g_{11}\,xx' + g_{12}\,yy' + g_{33}\,zz' + g_{12}(xy' + x'y) + g_{23}(yz' + y'z) + g_{31}(zx' + z'x),$$

where $g_{ij} = \mathbf{e}_i\,\mathbf{e}_j$ (the *fundamental tensor of the basis* $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$).

A plane is said to be *oriented* if a basis $\mathbf{e}_1$, $\mathbf{e}_2$ (with a fixed order of vectors) is introduced in the plane and the whole set of pairs $\mathbf{a}$, $\mathbf{b}$ of noncollinear vectors is partitioned into two subsets in the following manner. If an ordered pair $\mathbf{a}$, $\mathbf{b}$ of noncollinear vectors $\mathbf{a}$ and $\mathbf{b}$ have the same orientation as that of the basis $\mathbf{e}_1$, $\mathbf{e}_2$;

$$\mathbf{a}, \mathbf{b} \uparrow\uparrow \mathbf{e}_1, \mathbf{e}_2,$$

that is,

$$\overrightarrow{OAB} \uparrow\uparrow \overrightarrow{OE_1E_2}, \text{ where } \overrightarrow{OA} = \mathbf{a},\ \overrightarrow{OB} = \mathbf{b},\ \overrightarrow{OE_1} = \mathbf{e}_1,\ \overrightarrow{OE_2} = \mathbf{e}_2,$$

then this ordered pair $\mathbf{a}$, $\mathbf{b}$ is called the *right-hand* pair, or the *pair with positive orientation*. But if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{e}_1$, $\mathbf{e}_2$ have opposite orientations,

$$\mathbf{a}, \mathbf{b} \uparrow\downarrow \mathbf{e}_1, \mathbf{e}_2,$$

that is, (with notation the same as above) $\overrightarrow{OAB} \uparrow\downarrow \overrightarrow{OE_1E_2}$, then the ordered pair $\mathbf{a}$, $\mathbf{b}$ is called the *left-hand* pair, or the *pair with negative orientation*.

Suppose the noncollinear vectors $\mathbf{a}$ and $\mathbf{b}$ lie in an oriented plane. The *cross* (or *pseudoscalar*) *product* $(\mathbf{a}, \mathbf{b})$ of vector $\mathbf{a}$ and vector $\mathbf{b}$ is a number whose absolute value is equal to the area of a parallelogram with sides $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and which is positive if the ordered pair $\mathbf{a}$, $\mathbf{b}$ is a right-hand pair, and negative if that pair is a left-hand pair. If $\mathbf{a} \parallel \mathbf{b}$, then, by definition, $\mathbf{ab} = 0$.

The properties of a cross (or pseudoscalar) product are:

$$(\mathbf{a},\ \mathbf{b}) = -\,(\mathbf{b}, \mathbf{a}),$$

$$(\mathbf{a},\ \mathbf{b} + \mathbf{c}) = (\mathbf{a}, \mathbf{b}) + (\mathbf{a}, \mathbf{c}),$$

$$(\lambda\,\mathbf{a}, \mathbf{b}) = \lambda(\mathbf{a}, \mathbf{b}).$$

Suppose, in an oriented plane, there are given two noncollinear vectors $\mathbf{a}$ and $\mathbf{b}$, the angle between them being equal to α. The *angle φ formed by vector* $\mathbf{a}$ *and vector* $\mathbf{b}$ is the angle $\varphi = \alpha$ if $\mathbf{a}, \mathbf{b}$ is a right-hand pair, and the angle $\varphi = -\alpha$ if $\mathbf{a}, \mathbf{b}$ is a left-hand pair.

If the vectors $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors and $\mathbf{a} \uparrow\uparrow \mathbf{b}$, then, by definition. $\varphi = 0$, and if $\mathbf{a} \uparrow\downarrow \mathbf{b}$, then, by definition, $\varphi = \pi$. The angle φ from vector $\mathbf{a} \neq \mathbf{0}$ to vector $\mathbf{b} \neq \mathbf{0}$ is found from the relations

$$\cos \varphi = \frac{\mathbf{ab}}{|\mathbf{a}|\ |\mathbf{b}|}, \qquad \sin \varphi = \frac{(\mathbf{a}, \mathbf{b})}{|\mathbf{a}|\ |\mathbf{b}|}.$$

If the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are given by their components

$$\mathbf{a} = \{x, y\}, \qquad \mathbf{b} = \{x', y'\}$$

in the orthonormal basis $\mathbf{i}, \mathbf{j}$, then

$$\cos \varphi = \frac{xx' + yy'}{\sqrt{x^2 + y^2}\sqrt{x'^2 + y'^2}}, \quad \sin \varphi = \frac{xy' - x'y}{\sqrt{x^2 + y^2}\sqrt{x'^2 + y'^2}}.$$

If the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are given by their components in an arbitrary basis $\mathbf{e}_1, \mathbf{e}_2$, then

$$\cos \varphi = \frac{g_{11}xx' + g_{12}(xy' + x'y) + g_{22}yy'}{\sqrt{g_{11}x^2 + 2g_{12}xy + g_{22}y^2}\sqrt{g_{11}x'^2 + 2g_{12}x'y' + g_{22}y'^2}},$$

$$\sin \varphi = \frac{\sqrt{g}\,(xy' - x'y)}{\sqrt{g_{11}x^2 + 2g_{12}xy + g_{22}y^2}\sqrt{g_{11}x'^2 + 2g_{12}x'y' + g_{22}y'^2}},$$

where g is the *Gram determinant*:

$$g = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = (\mathbf{e}_1\, \mathbf{e}_2)^2.$$

Our formulas specify a set of values of the angle φ of the form $\{\varphi + 2k\pi\}$, where k assumes all integral values and φ is some value of the angle formed by vector $\mathbf{a}$ and vector $\mathbf{b}$.

Suppose vector $\mathbf{a}$ lies in a plane oriented by the basis $\mathbf{e}_1, \mathbf{e}_2$. The vector $\mathbf{a}_\varphi$ is a vector obtained by rotation of vector $\mathbf{a}$ through the angle φ (if $0 < \varphi < \pi$, then the pair $\mathbf{a}, \mathbf{a}_\varphi$ is a right-hand pair; and if $-\pi < \varphi < 0$, then it is a left-hand pair; if $\varphi = 0$, then $\mathbf{a}_\varphi = \mathbf{a}$, if $\varphi = \pi$, then $\mathbf{a}_\varphi = -\mathbf{a}$).

The vector obtained by rotation of vector $\mathbf{a}$ through the angle $\pi/2$ is symbolized thus: $[\mathbf{a}]$.

We have the following formula:

$$(\mathbf{a}, \mathbf{b}) = [\mathbf{a}]\,\mathbf{b}.$$

If, in the orthonormal basis $\mathbf{i}, \mathbf{j}$,

$$\mathbf{a} = \{x, y\}, \quad \mathbf{b} = \{x', y'\},$$

then

$$\mathbf{a}_\varphi = \{x \cos \varphi - y \sin \varphi,\ x \sin \varphi + y \cos \varphi\},$$

$$[\mathbf{a}] = \{-y, x\},$$

$$(\mathbf{a}, \mathbf{b}) = \begin{vmatrix} x & y \\ x' & y' \end{vmatrix}.$$

If the vectors $\mathbf{a} = \{x, y\}$ and $\mathbf{b} = \{x', y'\}$ are specified in an arbitrary basis $\mathbf{e}_1, \mathbf{e}_2$, then

$$(\mathbf{a}, \mathbf{b}) = \sqrt{g} \begin{vmatrix} x & y \\ x' & y' \end{vmatrix}.$$

Two bases $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}^1, \mathbf{e}^2$ in a plane are said to be *dual* bases if

$$\mathbf{e}_i \mathbf{e}^j = \delta_i^j = \begin{cases} 1 & \text{when } i = j, \\ 0 & \text{when } i \neq j. \end{cases}$$

The vectors of dual bases are connected by the relations

$$\mathbf{e}^1 = \frac{[\mathbf{e}_2]}{(\mathbf{e}_2, \mathbf{e}_1)}, \quad \mathbf{e}^2 = \frac{[\mathbf{e}_1]}{(\mathbf{e}_1, \mathbf{e}_2)}, \quad \mathbf{e}_1 = \frac{[\mathbf{e}^2]}{(\mathbf{e}^2, \mathbf{e}^1)}, \quad \mathbf{e}_2 = \frac{[\mathbf{e}^1]}{(\mathbf{e}^1, \mathbf{e}^2)}.$$

In other notation: the vectors of dual bases $\mathbf{a}, \mathbf{b}$ and $\mathbf{a}^*, \mathbf{b}^*$ are connected by the relations

$$(\mathbf{a}, \mathbf{a}^*) = (\mathbf{a}, \mathbf{b}^*) = 1, \quad (\mathbf{a}, \mathbf{b}^*) = (\mathbf{a}^*, \mathbf{b}) = 0,$$

$$\mathbf{a}^* = \frac{[\mathbf{b}]}{[\mathbf{b}, \mathbf{a}]}, \quad \mathbf{b}^* = \frac{[\mathbf{a}]}{(\mathbf{a}, \mathbf{b})}, \quad \mathbf{a} = \frac{[\mathbf{b}^*]}{(\mathbf{b}^*, \mathbf{a}^*)}, \quad \mathbf{b} = \frac{[\mathbf{a}^*]}{(\mathbf{a}^*, \mathbf{b}^*)}.$$

An expansion of an arbitrary vector $\mathbf{a}$ in terms of the bases $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}^1, \mathbf{e}^2$ is of the form

$$\mathbf{a} = (\mathbf{a}\mathbf{e}^1)\cdot\mathbf{e}_1 + (\mathbf{a}\mathbf{e}^2)\cdot\mathbf{e}_2,$$

$$\mathbf{a} = (\mathbf{a}\mathbf{e}_1)\cdot\mathbf{e}^1 + (\mathbf{a}\mathbf{e}_2)\cdot\mathbf{e}^2,$$

where in parentheses we have scalar products of the vectors *(Gibbs' formulas)*.

The numbers $a^1 = \mathbf{a}\mathbf{e}^1$, $a^2 = \mathbf{a}\mathbf{e}^2$, that is, the coefficients of $\mathbf{e}_1$ and $\mathbf{e}_2$ in the expansion of the vector $\mathbf{a}$ in terms of the vectors $\mathbf{e}_1, \mathbf{e}_2$ are called the *contravariant components* of the vector $\mathbf{a}$ with respect to the basis $\mathbf{e}_1, \mathbf{e}_2$, and the numbers $a_1 = \mathbf{a}\mathbf{e}_1$, $a_2 = \mathbf{a}\mathbf{e}_2$, that is, the coefficients of $\mathbf{e}^1, \mathbf{e}^2$ in the exapansion of vector $\mathbf{a}$ in terms of the basis $\mathbf{e}^1, \mathbf{e}^2$, are called the *covariant components* of the vector $\mathbf{a}$ with respect to the basis $\mathbf{e}_1, \mathbf{e}_2$.

If one of the vectors is specified in the basis $\mathbf{e}_1, \mathbf{e}_2$ by its contravariant components,

$$\mathbf{a} = x\mathbf{e}_1 + y\mathbf{e}_2 = \{x, y\},$$

and the other by its covariant components,

$$\mathbf{b} = x'\,\mathbf{e}^1 + y'\,\mathbf{e}^2 = [x', y'],$$

then

$$\mathbf{ab} = xx' + yy',$$

$$[\mathbf{a}] = \sqrt{g}\,[-y, x],$$

$$\mathbf{a}_\varphi = [xg_{11}\cos\varphi + y(g_{12}\cos\varphi - \sqrt{g}\sin\varphi),$$
$$x(g_{12}\cos\varphi + \sqrt{g}\sin\varphi) + yg_{22}\cos\varphi]$$

(when $\varphi = \pi/2$ we get the preceding formula).

A space is said to be *oriented* if a basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ has been introduced and the set of triples of noncoplanar vectors has been partitioned into two subsets. If an ordered triple $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ of noncoplanar vectors has the same orientation as basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, then it is termed a *right-hand triple*, or a *triple with positive orientation*. If the ordered triples $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ have opposite orientations, then the triple $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is termed a *left-hand triple*, or a *triple with negative orientation*.

The *triple scalar product* $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ of an ordered triple $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ of noncoplanar vectors lying in an oriented space is a number whose absolute value is equal to the volume of a parallelepiped with edges $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$ (O is an arbitrary point) and which is positive if the triple $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is right-hand, and negative if the triple $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is left-hand. If the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are coplanar, then, by definition, $(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 0$.

The properties of a triple scalar product of three vectors are:

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{b}, \mathbf{c}, \mathbf{a}) = (\mathbf{c}, \mathbf{a}, \mathbf{b}) = -(\mathbf{b}, \mathbf{a}, \mathbf{c}),$$

$$(\lambda\mathbf{a} + \mu\mathbf{b}, \mathbf{c}, \mathbf{d}) = \lambda(\mathbf{a}, \mathbf{c}, \mathbf{d}) + \mu(\mathbf{b}, \mathbf{c}, \mathbf{d})$$

(and analogously for the second and third factors).

If the space is oriented by means of the orthonormal basis $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and, in this basis,

$$\mathbf{a} = \{x, y, z\},\quad \mathbf{b} = \{x', y', z'\},\quad \mathbf{c} = \{x'', y'', z''\},$$

then

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}$$

and, in an arbitrary basis,

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \sqrt{g}\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix},$$

where g is the *Gram determinant*

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)^2.$$

Suppose two noncollinear vectors $\mathbf{a}$ and $\mathbf{b}$ are specified in an oriented space.

The *vector product* $[\mathbf{a}, \mathbf{b}]$ of vector $\mathbf{a}$ by vector $\mathbf{b}$ is a vector defined by the following conditions:

(1) $|[\mathbf{a}, \mathbf{b}]| = |\mathbf{a}|\,|\mathbf{b}| \sin \varphi$, where φ is the angle formed by vector $\mathbf{a}$ and vector $\mathbf{b}$;

(2) $[\mathbf{a}, \mathbf{b}] \perp \mathbf{a}$, $[\mathbf{a}, \mathbf{b}] \perp \mathbf{b}$;

(3) the ordered triple $\mathbf{a}$, $\mathbf{b}$ $[\mathbf{a}, \mathbf{b}]$ is a right-hand triple.

If $\mathbf{a} \parallel \mathbf{b}$, then, by definition, $[\mathbf{a}, \mathbf{b}] = \mathbf{0}$.

A vector product has the following properties:

$$[\mathbf{a}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}],$$

$$[\lambda\mathbf{a}, \mathbf{b}] = \lambda[\mathbf{a}, \mathbf{b}],$$

$$[\mathbf{a}, \mathbf{b} + \mathbf{c}] = [\mathbf{a}, \mathbf{b}] + [\mathbf{a}, \mathbf{c}],$$

$$[\mathbf{a}, \mathbf{b}]\,\mathbf{c} = \mathbf{a}[\mathbf{b}, \mathbf{c}] = (\mathbf{a}, \mathbf{b}, \mathbf{c}).$$

If, in an orthonormal basis $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$,

$$\mathbf{a} = \{x, y, z\}, \qquad \mathbf{b} = \{x', y', z'\},$$

then

$$[\mathbf{a}, \mathbf{b}] = \left\{ \begin{vmatrix} y & z \\ y' & z' \end{vmatrix}, \begin{vmatrix} z & x \\ z' & x' \end{vmatrix}, \begin{vmatrix} x & y \\ x' & y' \end{vmatrix} \right\}.$$

Two bases $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ are said to be *dual* bases if

$$\mathbf{e}_i\,\mathbf{e}^j = \delta_i^j = \begin{cases} 1 & \text{when } i = j, \\ 0 & \text{when } i \neq j. \end{cases}$$

The vectors of dual bases are connected by the relations

$$\mathbf{e}^1 = \frac{[\mathbf{e}_2\,\mathbf{e}_3]}{(\mathbf{e}_1\,\mathbf{e}_2\,\mathbf{e}_3)}, \quad \mathbf{e}^2 = \frac{[\mathbf{e}_3\,\mathbf{e}_1]}{[\mathbf{e}_1\,\mathbf{e}_2\,\mathbf{e}_3)}, \quad \mathbf{e}^3 = \frac{[\mathbf{e}_1\,\mathbf{e}_2]}{(\mathbf{e}_1\,\mathbf{e}_2\,\mathbf{e}_3)},$$

$$\mathbf{e}_1 = \frac{[\mathbf{e}^2\,\mathbf{e}^3]}{(\mathbf{e}^1\,\mathbf{e}^2\,\mathbf{e}^3)}, \quad \mathbf{e}_2 = \frac{[\mathbf{e}^3\,\mathbf{e}^1]}{(\mathbf{e}^1\,\mathbf{e}^2\,\mathbf{e}^3)}, \quad \mathbf{e}_3 = \frac{[\mathbf{e}^1\,\mathbf{e}^2]}{(\mathbf{e}^1\,\mathbf{e}^2\,\mathbf{e}^3)}.$$

Other notation: the dual bases $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ are defined by the relations

$$(\mathbf{a}, \mathbf{a}^*) = (\mathbf{b}, \mathbf{b}^*) = (\mathbf{c}, \mathbf{c}^*) = 1,$$

$$(\mathbf{a}, \mathbf{b}^*) = (\mathbf{a}^*, \mathbf{b}) = (\mathbf{b}, \mathbf{c}^*) = (\mathbf{b}^*, \mathbf{c}) = (\mathbf{c}, \mathbf{a}^*) = (\mathbf{a}, \mathbf{c}^*) = 0$$

and, furthermore,

$$\mathbf{a}^* = \frac{[\mathbf{bc}]}{(\mathbf{abc})}, \quad \mathbf{b}^* = \frac{[\mathbf{ca}]}{(\mathbf{abc})}, \quad \mathbf{c}^* = \frac{[\mathbf{ab}]}{(\mathbf{abc})},$$

$$\mathbf{a} = \frac{[\mathbf{b}^*\,\mathbf{c}^*]}{(\mathbf{a}^*\,\mathbf{b}^*\,\mathbf{c}^*)}, \quad \mathbf{b} = \frac{[\mathbf{c}^*\,\mathbf{a}^*]}{(\mathbf{a}^*\,\mathbf{b}^*\,\mathbf{c}^*)}, \quad \mathbf{c} = \frac{[\mathbf{a}^*\,\mathbf{b}^*]}{(\mathbf{a}^*\,\mathbf{b}^*\,\mathbf{c}^*)}.$$

Any vector $\mathbf{a}$ is expanded in terms of the bases $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ as follows:

$$\mathbf{a} = (\mathbf{ae}^1)\cdot\mathbf{e}_1 + (\mathbf{ae}^2)\cdot\mathbf{e}_2 + (\mathbf{ae}^3)\cdot\mathbf{e}_3,$$

$$\mathbf{a} = (\mathbf{ae}_1)\cdot\mathbf{e}^1 + (\mathbf{ae}_2)\cdot\mathbf{e}^2 + (\mathbf{ae}_3)\cdot\mathbf{e}^3,$$

where the numbers in parentheses are scalar products of vectors. These are the *Gibbs formulas.*

The numbers $a^i = \mathbf{ae}^i$ are called the *contravariant components (coordinates) of the vector* $\mathbf{a}$ in the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and the numbers $a_i = \mathbf{ae}_i$ are called the *covariant components (coordinates) of the vector* $\mathbf{a}$ in the same basis.

If the vectors $\mathbf{a}$ and $\mathbf{b}$ are specified by contravariant components in an arbitrary basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$,

$$\mathbf{a} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 = \{x, y, z\},$$

$$\mathbf{b} = x'\mathbf{e}_1 + y'\mathbf{e}_2 + z'\mathbf{e}_3 = \{x', y', z'\},$$

then the covariant components of the vector product

$$[\mathbf{a}, \mathbf{b}] = \sqrt{g}\left[\begin{vmatrix} y & z \\ y' & z' \end{vmatrix}, \begin{vmatrix} z & x \\ z' & x' \end{vmatrix}, \begin{vmatrix} x & y \\ x' & y' \end{vmatrix}\right].$$

We note here a number of other identities:

$$(\mathbf{a}, \mathbf{b})\,(\mathbf{c}, \mathbf{d}) = \begin{vmatrix} \mathbf{ac} & \mathbf{ad} \\ \mathbf{bc} & \mathbf{bd} \end{vmatrix}, [\mathbf{a}, \mathbf{b}]\,[\mathbf{c}, \mathbf{d}] = \begin{vmatrix} \mathbf{ac} & \mathbf{ad} \\ \mathbf{bc} & \mathbf{bd} \end{vmatrix},$$

$$[\mathbf{a}[\mathbf{b}, \mathbf{c}]] = \mathbf{b}(\mathbf{ac}) - \mathbf{c}\,(\mathbf{ab}), \ [\mathbf{a}, \mathbf{b}]^2 + (\mathbf{a}\,\mathbf{b})^2 = \mathbf{a}^2\mathbf{b}^2$$

(the parentheses in the last two formulas contain scalar products);

$$[[\mathbf{a}, \mathbf{b}]\,[\mathbf{c}, \mathbf{d}]] = \mathbf{c}(\mathbf{a}, \mathbf{b}, \mathbf{d}) - \mathbf{d}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \mathbf{b}\,(\mathbf{a}, \mathbf{c}, \mathbf{d}) - \mathbf{a}(\mathbf{b}, \mathbf{c}, \mathbf{d}),$$

$$\mathbf{a}(\mathbf{b}, \mathbf{c}, \mathbf{d}) + \mathbf{b}(\mathbf{c}, \mathbf{a}, \mathbf{d}) + \mathbf{c}(\mathbf{b}, \mathbf{d}, \mathbf{a}) + \mathbf{d}(\mathbf{a}, \mathbf{c}, \mathbf{b}) = 0,$$

$$[\mathbf{a}\,[\mathbf{b}, \mathbf{c}]] + [\mathbf{b}\,[\mathbf{c}, \mathbf{a}]] + [\mathbf{c}\,[\mathbf{a}, \mathbf{b}]] = 0,$$

$$(\mathbf{a}, \mathbf{b}, \mathbf{c})\,(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \begin{vmatrix} \mathbf{ax} & \mathbf{ay} & \mathbf{az} \\ \mathbf{bx} & \mathbf{by} & \mathbf{bz} \\ \mathbf{cx} & \mathbf{cy} & \mathbf{cz} \end{vmatrix}$$

$$(\mathbf{a}, \mathbf{b}, \mathbf{c})^2 = \begin{vmatrix} (\mathbf{a}, \mathbf{a}) & (\mathbf{a}, \mathbf{b}) & (\mathbf{a}, \mathbf{c}) \\ (\mathbf{b}, \mathbf{a}) & (\mathbf{b}, \mathbf{b}) & (\mathbf{b}, \mathbf{c}) \\ (\mathbf{c}, \mathbf{a}) & (\mathbf{c}, \mathbf{b}) & (\mathbf{c}, \mathbf{c}) \end{vmatrix}$$

$$([\mathbf{a}, \mathbf{b}], [\mathbf{b}, \mathbf{c}], [\mathbf{c}, \mathbf{a}]) = (\mathbf{a}, \mathbf{b}, \mathbf{c})^2.$$

There are problems in this text that also involve what are known as sliding vectors.

A *sliding vector* is also a directed line segment; however, the equality of such vectors is defined as follows: two nonzero sliding vectors $\overrightarrow{AB}$ an $\overrightarrow{CD}$ are said to be equal if the lengths of segments AB and CD are equal, if $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are in the same direction, and if they lie on the same straight line. A sliding vector $\boldsymbol{\omega}$ is given in a plane by its components x, y and by the moment $\mathbf{z} = (\mathbf{r}, \boldsymbol{\omega})$ with respect to some point O ($\mathbf{r} = \overrightarrow{OM}$, where M is an arbitrary point of the straight line); $(\mathbf{r}, \boldsymbol{\omega})$ is the cross (or pseudoscalar) product.

Sec. 3. Analytic geometry

Let us fix a point O in the plane and a basis $\mathbf{e}_1$, $\mathbf{e}_2$. Lay off vectors $\mathbf{e}_1$, $\mathbf{e}_2$ from the point O:

$$\overrightarrow{OE_1} = \mathbf{e}_1, \qquad \overrightarrow{OE_2} = \mathbf{e}_2.$$

The collection of straight lines $Ox = OE_1$, $Oy = OE_2$ on which vectors $\mathbf{e}_1$, $\mathbf{e}_2$ of the basis are laid off from O is termed a *general Cartesian system of coordinates in the plane*. The line Ox is the *axis of abscissas*, or *x-axis*, and the line Oy is the *axis of ordinates*, or the *y-axis*. The point O is the *origin of coordinates*; Ox and Oy are the *axes of the coordinate system*.

A general Cartesian system of coordinates $Oxyz$ in space is defined in similar fashion.

If an orthonormal basis $\mathbf{i}$, $\mathbf{j}$ (respectively $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) is introduced in a plane (respectively in space) and a point O is fixed, then the corresponding system of coordinates is said to be *rectangular Cartesian*.

The *radius vector* $\mathbf{r}$ of point M is a directed line segment:

$$\mathbf{r} = \overrightarrow{OM}.$$

The *general Cartesian coordinates* x, y *of a point* M in the general Cartesian system of coordinates $(O, \mathbf{e}_1, \mathbf{e}_2)$ in the plane are the coordinates of its radius vector $\mathbf{r} = \overrightarrow{OM}$ in the basis $\mathbf{e}_1$, $\mathbf{e}_2$:

$$\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2.$$

The *coordinates of point* M in the rectangular Cartesian system of coordinates $(O, \mathbf{i}, \mathbf{j})$ in the plane are the coordinates x, y of its radius vector $\mathbf{r} = \overrightarrow{OM}$ in the basis $\mathbf{i}, \mathbf{j}$:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}.$$

Similarly defined are the general Cartesian and rectangular Cartesian coordinates x, y, z of point M in space:

$$\mathbf{r} = \overrightarrow{OM} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3,$$

$$\mathbf{r} = \overrightarrow{OM} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

If the point M has coordinates x, y, then we write $M(x, y)$ or $M = (x, y)$ [in space: $M(x, y, z)$ or $M = (x, y, z)$].

If $\mathbf{r}_1$ and $\mathbf{r}_2$ are the radius vectors of points A and B, then

$$\overrightarrow{AB} = \mathbf{r}_2 - \mathbf{r}_1.$$

In coordinates:

$$\overrightarrow{AB} = \{x_2 - x_1, \quad y_2 - y_1\} \qquad \text{(in the plane)},$$

$$\overrightarrow{AB} = \{x^2 - x_1, \quad y_2 - y_1, \quad z_2 - z_1\} \qquad \text{(in space)},$$

where $A = (x_1, y_1)$, $B = (x_2, y_2)$ (in the plane); $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$ (in space).

The *length d of a line segment AB* whose endpoints are specified by radius vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are computed from the formula

$$d = |\mathbf{r}_2 - \mathbf{r}_1|.$$

In a rectangular Cartesian system of coordinates:

$$d = \sqrt{(x_2 - x_1)^2 (y_2 - y_1)^2} \qquad \text{(in the plane)},$$

$$d = \sqrt{(x_2 - x_1)^2 (y_2 - y_1)^2 + (z_2 - z_1)^2} \qquad \text{(in space)}$$

In a general Cartesian system of coordinates:

$$d = \sqrt{g_{11} (x_2 - x_1)^2 + 2g_{12}(x_2 - x_1)(y_2 - y_1) + g_{22} (y_2 - y_1)^2} \quad \text{(in the plane)}$$

$$\begin{aligned} d^2 = {} & g_{11} (x_2 - x_1)^2 + g_{22} (y_2 - y_1)^2 + g_{33} (z_2 - z_1)^2 \\ & + 2g_{12} (x_2 - x_1)(y_2 - y_1) + 2g_{23} (y_2 - y_1)(z_2 - z_1) \\ & + 2g_{31} (z_2 - z_1)(x_2 - x_1) \quad \text{(in space)}, \end{aligned}$$

where g_{ij} are the components of the metric tensor introduced in Sec. 2.

In particular, the distance r from the point to the origin of coordinates in a rectangular Cartesian system of coordinates:

$$r = \sqrt{x^2 + y^2} \qquad \text{(in the plane)}$$

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \text{(in space)},$$

and in a general Cartesian system of coordinates:

$$r = \sqrt{g_{11}x^2 + 2g_{12}\,xy + g_{22}y^2} \qquad \text{(in the plane)}$$

$$r = \sqrt{g_{11}x^2 + g_{22}y^2 + g_{33}\,z^2 + 2g_{12}xy + 2g_{23}yz + g_{31}zx} \qquad \text{(in space).}$$

Suppose a rectangular Cartesian system of coordinates has been introduced in the plane. Let us consider an arbitrary point M different from the coordinate origin. The *first polar coordinate* of point M is the length of segment OM. The *second polar coordinate* of the point M is the angle φ formed by vector $\mathbf{i}$ and the radius vector $\overrightarrow{OM}$ of the point M.

If x, y are the coordinates of point M in a rectangular Cartesian system of coordinates, then

$$x = \rho \cos\varphi, \qquad y = \rho \sin\varphi;$$

$$\rho = \sqrt{x^2 + y^2}, \quad \cos\varphi = \frac{x}{\rho} = \frac{x}{\sqrt{x^2 + y^2}}, \quad \sin\varphi = \frac{y}{\rho} = \frac{y}{\sqrt{x^2 + y^2}}.$$

For the origin, by definition, we have $\rho = 0$ (φ any number).

The ratio λ in which the point $M \neq M_2$ divides the nonzero directed line segment $\overrightarrow{M_1M_2}$ is the number

$$\lambda = \frac{\overrightarrow{M_1M}}{\overrightarrow{MM_2}}.$$

No matter what the number $\lambda \neq -1$ and no matter what the nonzero directed line segment $\overrightarrow{M_1M_2}$, there is one and only one point M which divides the line segment M_1M_2 in the ratio λ.

If $\mathbf{r}_1$ and $\mathbf{r}_2$ are the radius vectors of the points M_1 and M_2 then the radius vector $\mathbf{r}$ of point M is defined by the relation

$$\mathbf{r} = \frac{\mathbf{r}_1 + \lambda\mathbf{r}_2}{1 + \lambda}.$$

In particular, the radius vector of the midpoint of the line segment is equal to the half-sum of the radius vectors of its endpoints:

$$\mathbf{r} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2}.$$

In a general Cartesian system of coordinates, the coordinates of the point M are expressed in terms of the coordinates of the points M_1 and M_2 by the relations

$$x = \frac{x_1 + \lambda x_2}{1 + \lambda}, \quad y = \frac{y_1 + \lambda y_2}{1 + \lambda} \qquad \text{(in the plane),}$$

$$x = \frac{x_1 + \lambda x_2}{1 + \lambda}, \quad y = \frac{y_1 + \lambda y_2}{1 + \lambda}, \quad z = \frac{z_1 + \lambda z_2}{1 + \lambda} \qquad \text{(in space),}$$

and the coordinates of the midpoint of the line segment are:

$$x = \frac{x_1 + y_2}{2}, \qquad y = \frac{y_1 + y_2}{2} \qquad \text{(in the plane)},$$

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}, \quad z = \frac{z_1 + z_2}{2} \quad \text{(in space)}.$$

Points $A, B, C, \ldots$ are called *collinear* if there is a straight line on which they all lie.

Points $A, B, C, D, \ldots$ are said to be *coplanar* if there is a plane in which all the points lie.

If $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ are the radius vectors of the points A, B, C, then a necessary and sufficient condition for their collinearity is of the form

$$(\mathbf{r}_1 - \mathbf{r}_3, \mathbf{r}_2 - \mathbf{r}_3) = 0$$

and, in a general Cartesian system of coordinates,

$$\begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0,$$

where $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C = (x_3, y_3)$.

If $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4$ are the radius vectors of the points A, B, C, D, then a necessary and sufficient condition for their coplanarity is of the form

$$(\mathbf{r}_1 - \mathbf{r}_4, \mathbf{r}_2 - \mathbf{r}_4, \mathbf{r}_3 - \mathbf{r}_4) = 0,$$

and, in a general Cartesian system of coordinates,

$$\begin{vmatrix} x_1 - x_4 & y_1 - y_4 & z_1 - z_4 \\ x_2 - x_4 & y_2 - y_4 & z_2 - z_4 \\ x_3 - x_4 & y_3 - y_4 & z_3 - z_4 \end{vmatrix} = 0,$$

where

$$A = (x_1, y_1, z_1), \quad B = (x_2, y_2, z_2), \quad C = (x_3, y_3, z_3), \quad D = (x_4, y_4, z_4).$$

A *triangle* ABC is said to be a collection of three points A, B, C. An *oriented triangle* $\overrightarrow{ABC}$ is an ordered set of three points A, B, C. If the points A, B, C are noncollinear, then $\triangle\overrightarrow{ABC}$ is said to be *nondegenerate*, and if they are collinear, then the triangle is *degenerate*.

The area (ABC) of a nondegenerate *oriented* $\triangle\overrightarrow{ABC}$ lying in an oriented plane is a number whose absolute value is equal to the area of $\triangle ABC$, it is positive if $\triangle\overrightarrow{ABC}$ has a right-hand orientation (that is, the ordered pair $\overrightarrow{CA}, \overrightarrow{CB}$ has a right-hand orientation), and is negative if $\triangle\overrightarrow{ABC}$ has

a left-hand orientation. If the points A, B, C are collinear, then, by definition, we agree that $(ABC) = 0$.

The area (ABC) of the oriented $\triangle \overrightarrow{ABC}$ is computed from the formulas

$$(ABC) = \frac{1}{2}(\overrightarrow{CA}, \overrightarrow{CB}), \quad (ABC) = \frac{1}{2}(\mathbf{r}_1 - \mathbf{r}_3, \mathbf{r}_2 - \mathbf{r}_3)$$

($\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ are the radius vectors of the points A, B, C).

In a general Cartesian system of coordinates,

$$(ABC) = \frac{\sqrt{g}}{2}\begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{vmatrix} = \frac{\sqrt{9}}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

In a rectangular Cartesian system of coordinates

$$(ABC) = \frac{1}{2}\begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

To compute the area S of $\triangle ABC$, take the right-hand sides in absolute value:

$$S = \frac{\sqrt{g}}{2} \bmod \begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{vmatrix} = \frac{\sqrt{g}}{2} \bmod \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix},$$

$$S = \frac{1}{2} \bmod \begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{vmatrix} = \frac{1}{2} \bmod \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

In these formulas, $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C = (x_3, y_3)$.

The *ratio*

$$\frac{\overrightarrow{PQR}}{\overrightarrow{ABC}}$$

of the nondegenerate $\triangle \overrightarrow{PQR}$ and $\triangle \overrightarrow{ABC}$ lying in a plane is a number whose absolute value is equal to the ratio of the areas $\triangle PQR$ to the area of $\triangle ABC$ and which is positive if $\triangle \overrightarrow{PQR}$ and $\triangle \overrightarrow{ABC}$ have the same orientation (that is, $\overrightarrow{RP}, \overrightarrow{RQ} \uparrow\uparrow \overrightarrow{CA}, \overrightarrow{CB}$) and negative if $\triangle \overrightarrow{PQR}$ and $\triangle \overrightarrow{ABC}$ have opposite orientations. If $\overrightarrow{PQR}$ is a degenerate triangle and $\overrightarrow{ABC}$ is a nondegenerate triangle, then, by definition,

$$\frac{\overrightarrow{PQR}}{\overrightarrow{ABC}} = 0.$$

We have the formula

$$\frac{\overrightarrow{PQR}}{\overrightarrow{ABC}} = \frac{(PQR)}{(ABC)}$$

(if $\triangle\overrightarrow{PQR}$ and $\triangle\overrightarrow{ABC}$ lie in an oriented plane).

The *barycentric coordinates* α, β, γ of a point M with respect to a nondegenerate oriented triangle are the numbers

$$\alpha = \frac{\overrightarrow{MBC}}{\overrightarrow{ABC}}, \quad \beta = \frac{\overrightarrow{AMC}}{\overrightarrow{ABC}}, \quad \gamma = \frac{\overrightarrow{ABM}}{\overrightarrow{ABC}}.$$

A *tetrahedron* $ABCD$ is a set of four points A, B, C, D in space. An *oriented tetrahedron* $\overrightarrow{ABCD}$ is an ordered set of four points A, B, C, D in space. If the points A, B, C, D are noncoplanar, then the tetrahedron $\overrightarrow{ABCD}$ is said to be *nondegenerate* and if they are coplanar, it is said to be *degenerate*.

If a nondegenerate tetrahedron $\overrightarrow{ABCD}$ lies in an oriented space, then it has a *right-hand*, or *positive*, orientation if ordered triples of vectors $\overrightarrow{DA}$, $\overrightarrow{DB}$, $\overrightarrow{DC}$ and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ have the same orientation. But if these ordered triples of vectors have opposite orientations, then the tetrahedron $\overrightarrow{ABCD}$ has a *left-hand*, or *negative*, orientation.

The *volume* $(ABCD)$ of a nondegenerate oriented tetrahedron $\overrightarrow{ABCD}$ lying in an oriented space is a number whose absolute value is equal to the volume of the tetrahedron $ABCD$ and which is positive if the tetrahedron $\overrightarrow{ABCD}$ has a right-hand orientation and is negative if the orientation is left-hand. If $\overrightarrow{ABCD}$ is a degenerate tetrahedron, then, by definition, we assume that $(ABCD) = 0$.

The volume $(ABCD)$ of the oriented tetrahedron $\overrightarrow{ABCD}$ lying in an oriented space is computed from the formula

$$(ABCD) = \frac{1}{6}(\overrightarrow{DA}, \overrightarrow{DB}, \overrightarrow{DC}).$$

If $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{r}_4$ are the radius vectors of the points A, B, C, D, then

$$(ABCD) = \frac{1}{6}(\mathbf{r}_1 - \mathbf{r}_4, \ \mathbf{r}_2 - \mathbf{r}_4, \ \mathbf{r}_3 - \mathbf{r}_4).$$

In a general Cartesian system of coordinates,

$$(ABCD) = \frac{\sqrt{g}}{6}\begin{vmatrix} x_1 - x_4 & y_1 - y_4 & z_1 - z_4 \\ x_2 - x_4 & y_2 - y_4 & z_2 - z_4 \\ x_3 - x_4 & y_3 - y_4 & z_3 - z_4 \end{vmatrix} = \frac{\sqrt{g}}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix},$$

and in a rectangular Cartesian coordinate system,

$$(ABCD) = \frac{1}{6}\begin{vmatrix} x_1 - x_4 & y_1 - y_4 & z_1 - z_4 \\ x_2 - x_4 & y_2 - y_4 & z_2 - z_4 \\ x_3 - x_4 & y_3 - y_4 & z_3 - z_4 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

In these formulas,

$A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$, $C = (x_3, y_3, z_3)$, $D = (x_4, y_4, z_4)$

The *ratio*

$$\frac{\overrightarrow{PQRS}}{\overrightarrow{ABCD}}$$

of a nondegenerate oriented *tetrahedron* $\overrightarrow{PQRS}$ to a nondegenerate oriented tetrahedron $\overrightarrow{ABCD}$ is a number whose absolute value is equal to the ratio of the volume of the tetrahedron $PQRS$ to the volume of the tetrahedron $ABCD$ and which is positive if $\overrightarrow{PQRS}$ and $\overrightarrow{ABCD}$ have the same orientation (that is, the ordered triples $\overrightarrow{SP}$, $\overrightarrow{SQ}$, $\overrightarrow{SR}$ and $\overrightarrow{DA}$, $\overrightarrow{DB}$, $\overrightarrow{DC}$ have the same orientation), and negative if $\overrightarrow{PQRS}$ and $\overrightarrow{ABCD}$ have opposite orientations. If $\overrightarrow{PQRS}$ is a degenerate tetrahedron and $\overrightarrow{ABCD}$ is a nondegenerate one, then, by definition, we assume that

$$\frac{\overrightarrow{PQRS}}{\overrightarrow{ABCD}} = 0.$$

If the tetrahedrons $\overrightarrow{PQRS}$ and $\overrightarrow{ABCD}$ lie in an oriented space, then

$$\frac{\overrightarrow{PQRS}}{\overrightarrow{ABCD}} = \frac{(PQRS)}{(ABCD)}.$$

The *barycentric coordinates* α, β, γ, δ of a point M with respect to a non-degenerate oriented tetrahedron $\overrightarrow{ABCD}$ are the numbers

$$\alpha = \frac{\overrightarrow{MBCD}}{\overrightarrow{ABCD}}, \quad \beta = \frac{\overrightarrow{AMCD}}{\overrightarrow{ABCD}}, \quad \gamma = \frac{\overrightarrow{ABMD}}{\overrightarrow{ABCD}}, \quad \delta = \frac{\overrightarrow{ABCM}}{\overrightarrow{ABCD}}.$$

The area S, in space, of $\triangle ABC$ whose vertices are given by the radius vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ is computed from the formula

$$S = \frac{1}{2}\sqrt{[(\mathbf{r}_1 - \mathbf{r}_3)(\mathbf{r}_2 - \mathbf{r}_3)]^2}.$$

In a rectangular Cartesian coordinate system:

$$S = \frac{1}{2}\sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2},$$

where

$$\Delta_1 = \begin{vmatrix} y_1 - y_2 & z_1 - z_3 \\ y_2 - y_3 & z_2 - z_3 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} z_1 - z_3 & x_1 - x_3 \\ z_2 - z_3 & x_2 - x_3 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{vmatrix}$$

and in a general Cartesian coordinate system:

$$S = \frac{\sqrt{g}}{2}\sqrt{g^{11}\Delta_1^2 + g^{22}\Delta_2^2 + g^{33}\Delta_3^2 + 2g^{12}\Delta_1\Delta_2 + 2g^{23}\Delta_2\Delta_3 + 2g^{31}\Delta_3\Delta_1}.$$

In these formulas,

$$A = (x_1, y_1, z_1), \quad B = (x_2, y_2, z_2), \quad C = (x_3, y_3, z_3), \quad g^{ij} = \mathbf{e}^i\mathbf{e}^j.$$

If we introduce a general Cartesian coordinate system in the plane, then the *equation of any straight line* lying in the plane is the first-degree equation

$$Ax + By + C = 0$$

and, conversely, any first-degree equation

$$Ax + By + C = 0$$

(where $A^2 + B^2 \neq 0$) in any general Cartesian coordinate system is an equation of that straight line.

The *direction vector* $\mathbf{a} = \overrightarrow{PQ}$ of a straight line p is any nonzero vector, collinear with that straight line (vector $\overrightarrow{PQ} \neq 0$ and the straight line p are said to be *collinear* if the straight lines PQ and p are collinear, that is either parallel or coincident; the zero vector is assumed to be collinear with any straight line).

For a straight line specified in a general Cartesian coordinate system by the equation

$$Ax + By + C = 0,$$

the vector $\mathbf{a} = \{-B, A\}$ is a direction vector.

A necessary and sufficient condition of the collinearity of a vector $\mathbf{a} = \{l, m\}$ and a straight line $Ax+By+C=0$ in a general Cartesian coordinate system is of the form

$$Al + Bm = 0.$$

For the coordinates of all points (x, y) lying to one side of the straight line given by the equation

$$Ax + By + C = 0$$

with respect to a general Cartesian coordinate system, the following inequality is valid:

$$Ax + By + C > 0$$

(positive half-plane); and for the coordinates x, y of all points (x, y) lying to the other side of that straight line, the followng inequality is valid:

$$Ax + By + C < 0$$

(negative half-plane).

The vector $\mathbf{n} = \{A, B\}$ is termed the *principal vector of the straight line* specified with respect to a general Cartesian coordinate system by the equation $Ax + By + C = 0$. If it is laid off from any point M_0 of the straight line in question, $\overrightarrow{M_0P} = \mathbf{n}$, then the point P will lie in the positive half-plane.

If A and B are regarded as covariant components of the vector, then the vector $[A, B]$ is said to be *normal* to the straight line specified with respect to a general Cartesian coordinate system by the equation $Ax + By + C = 0$.

In a rectangular Cartesian coordinate system, the principal vector $\mathbf{n} = \{A, B\}$ of a straight line $Ax + By + C = 0$ is the normal vector to that line.

The equation of a straight line passing through point M_1 determined by the radius vector $\mathbf{r}_1$ and having the direction vector $\mathbf{a}$ is of the form

$$(\mathbf{r} - \mathbf{r}_1, \mathbf{a}) = 0.$$

If in a general Cartesian coordinate system, $\mathbf{a} = \{l, m\}$, $M_1 = (x_1, y_1)$, then the last equation becomes

$$\begin{vmatrix} x - x_1 & y - y_1 \\ l & m \end{vmatrix} = 0$$

or

$$\frac{x - x_1}{l} = \frac{y - y_1}{m}$$

(if $l = 0$, then this notation is to be understood as $x - x_1 = 0$, and if $m = 0$, then as $y - y_1 = 0$).

If $l \neq 0$, the last equation may be rewritten in the form

$$y - y_1 = k(x - x_1),$$

where $k = m/l$ is termed the *slope of the straight line* (in a general Cartesian coordinate system). The slope of the straight line that passes through two points (x_1, y_1) and (x_2, y_2) and is noncollinear with the y-axis is computed from the formula

$$k = \frac{y_2 - y_1}{x_2 - x_1}$$

(in a general Cartesian coordinate system).

In a rectangular Cartesian coordinate system, the slope is

$$k = \tan \alpha,$$

where α is the angle from the x-axis to the straight line under consideration.

The equation of the straight line cutting the y-axis in the point (O, b) and having slope k is, in a general Cartesian coordinate system, of the form

$$y = kx + b.$$

The equation of a straight line that does not pass through the coordinate origin and cuts the axes in points $(a, 0)$ and $(0, b)$ is of the form

$$\frac{x}{a} + \frac{y}{b} = 1$$

(intercept form of the equation of a straight line).

If a straight line passes through a point M_1 given by the radius vector $\mathbf{r}_1$ and has the direction vector $\mathbf{a}$, then the radius vector $\mathbf{r}$ of any point of it can be represented as

$$\mathbf{r} = \mathbf{r}_1 + t\mathbf{a},$$

where t takes on all real values. This equation is called the *parametric equation of the straight line*. The number t is the *coordinate* of point M on the line in question provided that M_1 is the coordinate origin and $\mathbf{a}$ is the basis vector:

$$t = \frac{\mathbf{r} - \mathbf{r}_1}{\mathbf{a}} = \frac{\overrightarrow{M_1M}}{\mathbf{a}}.$$

Suppose in a general Cartesian coordinate system we have $\mathbf{r} = \{x, y\}$, $\mathbf{r}_1 = \{x_1, y_1\}$, $\mathbf{a} = \{l, m\}$; then we obtain the parametric equations of the straight line in the form

$$x = x_1 + lt, \qquad y = y_1 + mt.$$

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the radius vectors of two distinct points M_1 and M_2. Then the equation of the straight line M_1M_2 is of the form

$$(\mathbf{r} - \mathbf{r}_1, \mathbf{r}_2 - \mathbf{r}_1) = 0.$$

In a general Cartesian coordinate system,

$$\begin{vmatrix} x - x_1 & y - y_1 \\ x_2 - x_1 & y_2 - y_1 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

or

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

(if any one of the denominators is equal to zero, then this notation is to be understood in the sense that the numerator is also zero).

The equation of a straight line passing through point M_1 given by radius vector $\mathbf{r}_1$ and having the normal vector $\mathbf{n}$ is

$$\mathbf{n}(\mathbf{r} - \mathbf{r}_1) = 0.$$

If in a rectangular Cartesian coordinate system $\mathbf{r}_1 = \{x_1, y_1\}$, $\mathbf{n} = \{A, B\}$, $\mathbf{r}\{x, y\}$, then the equation $\mathbf{n}(\mathbf{r} - \mathbf{r}_1) = 0$ becomes

$$A(x - x_1) + B(y - y_1) = 0.$$

Such also is the form of the equation $\mathbf{n}(\mathbf{r} - \mathbf{r}_1) = 0$ in general Cartesian coordinates if $\mathbf{r}_1 = \{x_1, y_1\}$, $\mathbf{r} = \{x, y\}$ but $\mathbf{n} = [A, B]$ (covariant coordinates).

A necessary and sufficient condition that two straight lines specified by the equations

$$Ax + By + C = 0,$$

$$A'x + B'y + C' = 0$$

with respect to a general Cartesian coordinate system intersect, are parallel, or are coincident is that the system of equations have only one solution, no solution, or an infinity of solutions, respectively.

If these straight lines intersect, then to find the coordinates of their point of intersection we have to solve the following system of equations:

$$Ax + By + C = 0,$$

$$A'x + B'y + C' = 0.$$

A *pencil of straight lines* is the set of all straight lines passing through the same point S *(proper pencil*; the point S is called the *centre of the pencil)* or the set of all parallel straight lines *(improper pencil)*.

If general Cartesian coordinates are used to specify the equations of two straight lines

$$Ax + By + C = 0$$

$$A'x + B'y + C' = 0$$

that intersect in a point S, then the equation of the pencil with centre S is of the form

$$\alpha(Ax + By + C) + \beta(A'x + B'y + C') = 0,$$

where α and β take on all values with at least one nonzero value.

If the straight lines

$$Ax + By + C = 0,$$

$$A'x + B'y + C' = 0$$

are parallel, then the preceding equation is the equation or an improper pencil to which the given lines belong; note that all possible values are taken for α and β but such that at least one of the coefficients of x and y is nonzero:

$$\alpha A + \beta A', \qquad \alpha B + \beta B'.$$

A necessary and sufficient condition that three straight lines given in a general Cartesian coo rdinate system by the equations

$$Ax + By + C = 0,$$

$$A'x + B'y + C' = 0,$$

$$A''x + B''y + C'' = 0$$

belong to a single pencil is the equality

$$\begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix} = 0.$$

Here, the straight lines in question belong to a single proper pencil if at least one of the determinants

$$\begin{vmatrix} A' & B' \\ A'' & B'' \end{vmatrix}, \quad \begin{vmatrix} A'' & B'' \\ A & B \end{vmatrix}, \quad \begin{vmatrix} A & B \\ A' & B' \end{vmatrix}$$

is different from zero, and they belong to one improper pencil if all the determinants are equal to zero.

If in a rectangular Cartesian coordinate system two straight lines are given by the equations

$$Ax + By + C = 0,$$
$$A'x + B'y + C' = 0,$$

then the cosines of the angles $\varphi_{1,2}$ between them are determined from the formula

$$\cos\varphi_{1,2} = \pm\frac{AA' + BB'}{\sqrt{A^2 + B^2}\,\sqrt{A'^2 + B'^2}}.$$

In general Cartesian coordinates,

$$\cos\varphi_{1,2} = \pm\frac{g_{11}AA' + g_{12}(AB' + A'B) + g_{22}BB'}{\sqrt{g_{11}A^2 + 2g_{12}AB + g_{22}B^2}\,\sqrt{g_{11}A'^2 + 2g_{12}A'B' + g_{22}B'^2}}.$$

The angle φ formed by the straight line $Ax + By + C = 0$ and the straight line $A'x + B'y + C' = 0$ in rectangular Cartesian coordinates is found from the relations

$$\cos\varphi = \frac{AA' + BB'}{\sqrt{A^2 + B^2}\,\sqrt{A'^2 + B'^2}}, \quad \sin\varphi = \frac{AB' - A'B}{\sqrt{A^2 + B^2}\,\sqrt{A'^2 + B'^2}},$$

and if the straight lines are not perpendicular, then it suffices to know only the value of $\tan\varphi$:

$$\tan\varphi = \frac{AB' - A'B}{AA' + BB'}$$

In general Cartesian coordinates we have

$$\cos\varphi = \frac{g^{11}AA' + g^{12}(AB' + A'B) + g^{22}BB'}{\sqrt{g^{11}A^2 + 2g^{12}AB + g^{22}B^2}\,\sqrt{g^{11}A'^2 + 2g^{12}A'B' + g^{22}B'^2}}$$
$$= \frac{g_{22}AA' - 2g_{12}(AB' + A'B) + g_{11}BB'}{\sqrt{g_2A^2 - 2g_{12}AB + g_{11}B^2}\,\sqrt{g_{22}A'^2 - 2g_{12}A'B' + g_{11}B'^2}},$$

$$\sin\varphi = \frac{AB' - A'B}{\sqrt{g}\,\sqrt{g^{11}A^2 + 2g^{12}AB + g^{22}B^2}\,\sqrt{g^{11}A'^2 + 2g^{12}A'B' + g^{22}B'^2}}$$
$$= \frac{\sqrt{g}\,(AB' - A'B)}{\sqrt{g_{22}A^2 - 2g_{12}AB + g_{11}B^2}\,\sqrt{g_{22}A'^2 - 2g_{12}A'B' + g_{11}B'^2}},$$

and if the straight lines are not mutually perpendicular, then

$$\tan\varphi = \frac{AB' - A'B}{\sqrt{g}(g^{11}AA' + g^{12}(AB' + A'B) + g^{22}BB'}$$
$$= \frac{\sqrt{g}\,(AB' - A'B)}{g_{22}AA' - 2g_{12}(AB' + A'B) + g_{11}BB'}.$$

Ordinarily, for the angle formed by the straight line $Ax + By + C = 0$ and the straight line $A'x + B'y + C' = 0$ one takes the set of values $\{\varphi + k\pi\}$ (k assumes all integral values), where φ is one of the values of the angle formed by the straight lines.

If, in a rectangular Cartesian coordinate system, the straight lines p and p' have slopes k and k' and if they are not mutually perpendicular, then the tangents of the angles $\varphi_{1,2}$ between p and p' are computed from the formula

$$\tan \varphi_{1,2} = \pm \frac{k' - k}{1 + kk'},$$

and in a general Cartesian system of coordinates

$$\tan \varphi_{1,2} = \pm \frac{\sqrt{g}(k' - k)}{g_{11} - g_{12}(k + k') + g_{22}kk'}.$$

The tangent of the angle φ formed by straight line p and straight line p' is

$$\tan \varphi = \frac{k' - k}{1 + kk'},$$

and in a general Cartesian coordinate system

$$\tan \varphi = \frac{\sqrt{g}(k' - k)}{g_{11} - g_{12}(k + k') + g_{22}kk'}.$$

If the straight lines p and p' have, in general Cartesian coordinates, the slopes k and k', then a necessary and sufficient condition for their *collinearity* is that

$$k = k'.$$

A necessary and sufficient condition of *perpendicularity* is that

$$g_{11} - g_{12}(k + k') + g_{22}kk' = 0,$$

and, in rectangular Cartesian coordinates,

$$1 + kk' = 0 \text{ or } kk' = -1.$$

If a straight line is given by the equation $Ax + By + C = 0$ in rectangular Cartesian coordinates, then the distance d from point $M_0(x_0, y_0)$ to this straight line is computed from the formula

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}},$$

and if the vector $\mathbf{n} = \{A, B\}$, which is normal to the straight line under consideration, is a unit vector: $A^2 + B^2 = 1$ (in this case the equation $Ax + By + C = 0$ is said to be *normal*), then

$$d = |Ax_0 + By_0 + C|.$$

In general Cartesian coordinates,

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{g^{11}A^2 + 2g^{12}AB + g^{22}B^2}} = \sqrt{g}\,\frac{|Ax_0 + By_0 + C_j|}{\sqrt{g_{22}A^2 - 2g_{12}AB + g_{11}B^2}},$$

and if the vector $[A, B]$ is a unit vector, then

$$d = |Ax_0 + By_0 + C|.$$

If the straight line is given by the equation

$$(\mathbf{a}, \mathbf{r} - \mathbf{r}_1) = 0,$$

then the distance d from the point M_0 with radius vector $\mathbf{r}_0$ to this straight line is

$$d = \frac{|(\mathbf{a}, \mathbf{r}_1 - \mathbf{r}_0)|}{|\mathbf{a}|},$$

and if $\mathbf{a}$ is a unit vector, then

$$d = |(\mathbf{a}, \mathbf{r}_1 - \mathbf{r}_0)|.$$

If in rectangular Cartesian coordinates, $\mathbf{a} = \{l, m\}$, $\mathbf{r}_0 = \{x_0, y_0\}$, $\mathbf{r}_1 = \{x_1, y_1\}$, then

$$d = \frac{\operatorname{mod}\begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ l & m \end{vmatrix}}{\sqrt{l^2 + m^2}},$$

and if the vector $\mathbf{a}$ is a unit vector ($l^2 + m^2 = 1$), then

$$d = \operatorname{mod}\begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ l & m \end{vmatrix}$$

where the symbol $\operatorname{mod} x$ denotes the absolute value of the number x. In general Cartesian coordinates,

$$d = \frac{\sqrt{g}\operatorname{mod}\begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ l & m \end{vmatrix}}{\sqrt{g_{11}l^2 + 2g_{12}lm + g_{22}m^2}},$$

and if $\mathbf{a} = \{l, m\}$ is a unit vector, then

$$d = \sqrt{g}\operatorname{mod}\begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ l & m \end{vmatrix}.$$

If the straight line is given by the equation

$$\mathbf{n}(\mathbf{r} - \mathbf{r}_1) = 0,$$

then the distance d from the point M_0 with radius vector $\mathbf{r}_0$ to this straight line is computed from the formula

$$d = \frac{|\mathbf{n}(\mathbf{r}_1 - \mathbf{r}_0)|}{|\mathbf{n}|},$$

and if the vector $\mathbf{n}$ is a unit vector, then

$$d = |\mathbf{n}(\mathbf{r}_1 - \mathbf{r}_0)|.$$

If in rectangular Cartesian coordinates $\mathbf{r}_0 = \{x_0, y_0\}$, $\mathbf{r}_1 = \{x_1, y_1\}$, $n = \{A, B\}$, then

$$d = \frac{|A(x_1 - x_0) + B(y_1 - y_0)|}{\sqrt{A^2 + B^2}},$$

and if the vector $\{A, B\}$ is a unit vector, then

$$d = |A(x_0 - x_1) + B(y_0 - y_1)|.$$

In general Cartesian coordinates,

$$d = \frac{|g_{11}A(x_1 - x_0) + g_{12}[A(y_1 - y_0) + B(x_1 - x_0)] + g_{22}B(y_1 - y_0)|}{\sqrt{g_{11}A^2 + 2g_{12}AB + g_{22}B^2}}$$

and if $|\{A, B\}| = 1$, then

$$d = |g_{11}A(x_1 - x_0) + g_{12}[A(y_1 - y_0) + B(x_1 - x_0)] + g_{22}B(y_1 - y_0)|.$$

If the vector $\mathbf{n}$ is given by covariant components, $\mathbf{n} = [A, B]$, then

$$d = \frac{|A(x_1 - x_0) + B(y_1 - y_0)|}{\sqrt{g^{11}A^2 + 2g^{12}AB + g^{22}B^2}} = \sqrt{g}\,\frac{|A(x_1 - x_0) + B(y_1 - y_0)|}{\sqrt{g_{22}A^2 - 2g_{12}AB + g_{11}B^2}},$$

and if $|\mathbf{n}| = 1$, then

$$d = |A(x_1 - x_0) + B(y_1 - y_0)|.$$

The area (ABC) of an oriented $\triangle \overrightarrow{ABC}$ whose sides are specified with respect to a general Cartesian coordinate system by the equations

$$(BC)\colon\ A_1x + B_1y + C_1 = 0,$$

$$(CA)\colon\ A_2x + B_2y + C_2 = 0,$$

$$(AB)\colon\ A_3x + B_3y + C_3 = 0,$$

is computed from the formula

$$(ABC) = \frac{\sqrt{g}}{2}\,\frac{\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}^2}{\begin{vmatrix} A_2 & B_2 \\ A_3 & B_3 \end{vmatrix}\begin{vmatrix} A_3 & B_3 \\ A_1 & B_1 \end{vmatrix}\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}},$$

and, in rectangular Cartesian coordinates, from the formula

$$(ABC) = \frac{1}{2} \cdot \frac{\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}^2}{\begin{vmatrix} A_2 & B_2 \\ A_3 & B_3 \end{vmatrix} \begin{vmatrix} A_3 & B_3 \\ A_1 & B_1 \end{vmatrix} \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}.$$

The area S of $\triangle ABC$ is

$$S = \frac{\sqrt{g}}{2} \frac{\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}^2}{\operatorname{mod}\left(\begin{vmatrix} A_2 & B_2 \\ A_3 & B_3 \end{vmatrix} \begin{vmatrix} A_3 & B_3 \\ A_1 & B_1 \end{vmatrix} \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}\right)},$$

and, in rectangular Cartesian coordinates,

$$S = \frac{1}{2} \frac{\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}}{\operatorname{mod}\left(\begin{vmatrix} A_2 & B_2 \\ A_3 & B_3 \end{vmatrix} \begin{vmatrix} A_3 & B_3 \\ A_1 & B_1 \end{vmatrix} \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}\right)}.$$

If a general Cartesian coordinate system $Oxyz$ is introduced in space, then the equation of any plane in the first-degree equation

$$Ax + By + Cz + D = 0,$$

and, conversely, any first-degree equation

$$Ax + By + Cz + D = 0$$

(where $A^2 + B^2 + C^2 \neq 0$) in any general Cartesian system of coordinates is the equation of the plane.

For the coordinates of all points (x, y, z) lying to one side of a plane that is specified with respect to a general Cartesian system of coordinates by the equation

$$Ax + By + Cz + D = 0,$$

the following inequality is valid:

$$Ax + By + Cz + D > 0$$

(positive half-space), and for the coordinates of all points (x, y, z) lying to the other side of that plane, the following inequality is valid:

$$Ax + By + Cz + D < 0$$

(negative half-space).

The vector $\mathbf{n} = \{A, B, C\}$ is called the *principal vector of the plane* specified by the equation

$$Ax + By + Cz + D = 0.$$

If it is laid off from any point M_0 of the plane, $\overrightarrow{M_0P} = \mathbf{n}$, then point P lies in the positive half-space.

If A, B, C are regarded as covariant components, then the vector $[A, B, C]$ is normal to the plane

$$Ax + By + Cz + D = 0.$$

In a rectangular Cartesian system of coordinates, the principal vector $\mathbf{n} = \{A, B, C\}$ of the plane specified by the equation $Ax + By + Cz + D = 0$ is a vector normal to that plane.

The equation of a plane passing through point M_1 defined by the radius vector $\mathbf{r}_1$ and normal to the vector $\mathbf{n}$ is of the form

$$\mathbf{n}(\mathbf{r} - \mathbf{r}_1) = 0.$$

If in general Cartesian coordinates $\mathbf{r}_1 = \{x_1, y_1, z_1\}$, $\mathbf{n}[A, B, C]$, $\mathbf{r} = \{x, y, z\}$, then the equation $\mathbf{n}(\mathbf{r} - \mathbf{r}_1) = 0$ becomes

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0.$$

In rectangular Cartesian coordinates,

$$\mathbf{n} = \{A, B, C\} = [A, B, C].$$

The vector $\overrightarrow{PQ} \neq 0$ and the plane π are said to be *coplanar* if the straight line PQ is either parallel to, or lies, in the plane π. The zero vector is assumed to be coplanar with any plane.

For a vector $\mathbf{a} = \{l, m, n\}$ and a plane $Ax + By + Cz + D = 0$ to be coplanar, it is necessary and sufficient that the following equality be valid:

$$Al + Bm + Cn = 0$$

(the coordinates are general Cartesian).

The equation of a plane passing through a point $M_1(\mathbf{r}_1)$ and coplanar with two noncollinear vectors $\mathbf{a}$ and $\mathbf{b}$ is of the form

$$(\mathbf{r} - \mathbf{r}_1, \mathbf{a}, \mathbf{b}) = 0.$$

In general Cartesian coordinates,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0,$$

where $\mathbf{a} = \{l_1, m_1, n_1\}$, $\mathbf{b} = \{l_2, m_2, n_2\}$, $\mathbf{r}_1 = \{x_1, y_1, z_1\}$, $\mathbf{r} = \{x, y, z\}$.

The equation of a plane passing through two points $M_1(\mathbf{r}_1)$ and $M_2(\mathbf{r}_2)$ and coplanar with the vector $\mathbf{a} \nparallel \overrightarrow{M_1M_2}$ is of the form

$$(\mathbf{r} - \mathbf{r}_1,\ \mathbf{r}_2 - \mathbf{r}_1, \mathbf{a}) = 0.$$

In general Cartesian coordinates,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l & m & n \end{vmatrix} = 0,$$

where $M_1 = (x_1, y_1, z_1)$, $M_2 = (x_2, y_2, z_2)$, $\mathbf{a} = \{l, m, n\}$.

The equation of a plane passing through three noncollinear points $M_1(\mathbf{r}_1)$, $M_2(\mathbf{r}_2)$, $M_3(\mathbf{r}_3)$ has the form

$$(\mathbf{r} - \mathbf{r}_3,\ \mathbf{r}_1 - \mathbf{r}_3,\ \mathbf{r}_2 - \mathbf{r}_3) = 0.$$

and, in general Cartesian coordinates,

$$\begin{vmatrix} x - x_3 & y - y_3 & z - z_3 \\ x_1 - x_3 & y_1 - y_3 & z_1 - z_3 \\ x_2 - x_3 & y_2 - y_3 & z_2 - z_3 \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & x_3 & z_3 & 1 \end{vmatrix} = 0,$$

where $M_1 = (x_1, y_1, z_1)$, $M_2 = (x_2, y_2, z_2)$, $M_3 = (x_3, y_3, z_3)$.

The equation of a plane passing through a point $M_1(\mathbf{r}_1)$ (and coplanar with two noncollinear vectors $\mathbf{a}$ and $\mathbf{b}$ can be written in parametric form:

$$\mathbf{r} = \mathbf{r}_1 + u\mathbf{a} + v\mathbf{b}.$$

In this equation, u and v are the coordinates of the point M in a general Cartesian coordinate system in the plane at hand, in which the origin is the point $M_1(\mathbf{r}_1)$ and the basis is $\mathbf{a}, \mathbf{b}$.

The parametric equations of a plane in a general Cartesian coordinate system are:

$$x = x_1 + ul_1 + vl_2,$$

$$y = y_1 + um_1 + vm_2,$$

$$z = z_1 + un_1 + vn_2,$$

where $\mathbf{r}_1 = \{x_1, y_1, z_1\}$, $\mathbf{a} = \{l_1, m_1, n_1\}$, $\mathbf{b} = \{l_2, m_2, n_2\}$.

The parametric equation of a plane passing through points $M_1(\mathbf{r}_1)$, $M_2(\mathbf{r}_2)$ and coplanar with the vector $\mathbf{a} \nparallel \overrightarrow{M_1M_2}$ is of the form

$$\mathbf{r} = \mathbf{r}_1 + u\mathbf{a} + v(\mathbf{r}_2 - \mathbf{r}_1).$$

In general Cartesian coordinates,

$$x = x_1 + ul + v(x_2 - x_1),$$
$$y = y_1 + um + v(y_2 - y_1).$$
$$z = z_1 + un + v(z_2 - z_1),$$

where $M_1 = (x_1, y_1, z_1)$, $M_2 = (x_2, y_2, z_2)$, $\mathbf{a} = \{l, m, n\}$.

The parametric equation of a plane passing through three noncollinear points $M_1(\mathbf{r}_1)$, $M_2(\mathbf{r}_2)$, $M_3(\mathbf{r}_3)$ is of the form

$$\mathbf{r} = \mathbf{r}_3 + u(\mathbf{r}_1 - \mathbf{r}_3) + v(\mathbf{r}_2 - \mathbf{r}_3)$$

and, in general Cartesian coordinates,

$$x = x_3 + u(x_1 - x_3) + v(x_2 - x_3),$$
$$y = y_3 + u(y_1 - y_3) + v(y_2 - y_3),$$
$$z = z_3 + u(z_1 - z_3) + v(z_2 - z_3),$$

where $M_1 = (x_1, y_1, z_1)$ $M_2 = (x_2, y_2, z_2)$ $M = (x_3, y_3, z_3)$.

If the plane does not pass through the coordinate origin of a general Cartesian coordinate system and cuts the coordinate axes in the points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, then its equation can be written as

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

(the *intercept form of the equation of the plane*).

For two planes given by the equations

$$Ax + By + Cz + D = 0,$$
$$A'x + B'y + C'z + D' = 0$$

with respect to a general Cartesian coordinate system to intersect, it is necessary and sufficient that the vector

$$\mathbf{a} = \left\{\begin{vmatrix} B & C \\ B' & C' \end{vmatrix}, \quad \begin{vmatrix} C & A \\ C' & A' \end{vmatrix}, \quad \begin{vmatrix} A & B \\ A' & B' \end{vmatrix}\right\}$$

be nonzero: $\mathbf{a} \neq \mathbf{0}$. In this case ($\mathbf{a} \neq \mathbf{0}$), the vector $\mathbf{a}$ is the direction vector of the straight line along which the planes in question intersect.

For the two planes

$$Ax + By + Cz + D = 0,$$
$$A'x + B'y + C'z + D' = 0$$

to be parallel, it is necessary and sufficient that the vector **a** be equal to zero but that at least one of the determinants

$$\begin{vmatrix} A & D \\ A' & D' \end{vmatrix}, \quad \begin{vmatrix} B & D \\ B' & D' \end{vmatrix}, \quad \begin{vmatrix} C & D \\ C' & D' \end{vmatrix}$$

be different from zero.

For the two planes

$$Ax + By + Cz + D' = 0,$$
$$A'x + B'y + C'z + D' = 0$$

to be coincident, it is necessary and sufficient that the corresponding coefficients of the equations of the planes be proportional:

$$A' = kA, \quad B' = kB, \quad C' = kC, \quad D' = kD.$$

Three planes specified with respect to a general Cartesian coordinate system by the equations

$$Ax + By + Cz + D = 0,$$
$$A'x + B'y + C'z + D' = 0,$$
$$A''x + B''y + C''z + D'' = 0$$

have only one point in common if and only if the following inequality is valid:

$$\begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix} \neq 0.$$

To find the coordinates of this point it is necessary to solve the system of equations of the three given planes.

A *pencil of planes* is a set of all the planes passing through one straight line l *(proper pencil)*. The straight line l is called the *axis of the pencil*. An *improper pencil* of planes is the set of all parallel planes.

If two planes given by equations with respect to a general Cartesian system of coordinates,

$$Ax + By + Cz + D = 0,$$
$$A'x + B'y + C'z + D' = 0$$

intersect along a straight line l, then the equation of the pencil of planes with the axis l is of the form

$$\alpha(Ax + By + Cz + D) + \beta(A'x + B'y + C'z + D') = 0,$$

where α and β take on all possible values with at least one of them being different from zero.

If the planes are parallel, then the last equation is the equation of the improper pencil of planes to which they belong, and for α and β one takes all possible values with the exception of those for which all the coefficients of x, y, z, that is,

$$\alpha A + \beta A', \quad \alpha B + \beta B', \quad \alpha C + \beta C'$$

are equal to zero.

A *bundle* (or *sheaf) of planes* is the set of all planes that pass through the same point S *(proper bundle)* or the set of all planes coplanar with one straight line *(improper bundle)*. The point S is termed the *centre of the bundle.*

If three planes specified by equations in general Cartesian coordinates,

$$\begin{aligned} Ax + By + Cz + D &= 0, \\ A'x + B'y + C'z + D' &= 0, \\ A''x + B''y + C''z + D'' &= 0, \end{aligned}$$

have the same common point S, then the equation of the bundle of planes with centre S is of the form

$$\alpha(Ax + By + Cz + D) + \beta(A'x + B'y + C'z + D) + \gamma(A''x + B''y + C''z + D'') = 0,$$

where α, β, γ take on all values, at least one of which is nonzero.

If the three indicated planes are coplanar with one and the same straight line, but do not pass through one straight line, then the last equation is the equation of the *improper bundle* to which the three given planes belong; and for α, β, γ one takes all possible values, with the exception of those for which all the coefficients of x, y, z, that is,

$$\alpha A + \beta A' + \gamma A'', \quad \alpha B + \beta B' + \gamma B'', \quad \alpha C + \beta C' + \beta C''$$

are equal to zero.

The distance d from the point $M_0(x_0, y_0, z_0)$ to the plane given by the equation

$$Ax + By + Cz + D = 0$$

in rectangular Cartesian coordinates is computed from the formula

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

and if the vector $\mathbf{n} = \{A, B, C\}$ is the unit vector (the equation $Ax + By + Cz + D = 0$ is then said to be *normal)*, then

$$d = |Ax_0 + By_0 + Cz_0 + D|.$$

In general Cartesian coordinates,

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{g^{11}A^2 + g^{22}B^2 + g^{33}C^2 + 2g^{12}AB + 2g^{23}BC + 2g^{31}CA}}$$

$$= \frac{\sqrt{g}|Ax_0 + By_0 + Cz_0 + D|}{\left(- \begin{vmatrix} g_{11} & g_{12} & g_{13} & A \\ g_{21} & g_{22} & g_{23} & B \\ g_{31} & g_{32} & g_{33} & C \\ A & B & C & 0 \end{vmatrix} \right)^{1/2}},$$

and if the vector $[A, B, C]$ is the unit vector, then

$$d = |Ax_0 + By_0 + Cz_0 + D|.$$

The distance d from the point $M_0(\mathbf{r}_0)$ to the plane given by the equation $(\mathbf{n}, \mathbf{r} - \mathbf{r}_1) = 0$ is computed from the formula

$$d = \frac{|\mathbf{n}(\mathbf{r}_1 - \mathbf{r}_0)|}{|\mathbf{n}|},$$

and if $\mathbf{n}$ is the unit vector, then

$$d = |\mathbf{n}(\mathbf{r}_1 - \mathbf{r}_0)|.$$

If the vector $\mathbf{n}$ is given by covariant components in a general Cartesian system of coordinates: $\mathbf{n} = [A, B, C]$, and the vectors $\mathbf{r}_0$ and $\mathbf{r}_1$ are given by contravariant coordinates (components):

$$\mathbf{r}_0 = \{x_0, y_0, z_0\}, \qquad \mathbf{r}_1 = \{x_1, y_1, z_1\},$$

then

$$d = \frac{|A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)}{\sqrt{g^{11}A^2 + g^{22}B^2 + g^{33}C^2 + 2g^{12}AB + 2g^{23}BC + 2g^{31}CA}}$$

$$= \frac{\sqrt{g}|A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)|}{\left(- \begin{vmatrix} g_{11} & g_{12} & g_{13} & A \\ g_{21} & g_{22} & g_{23} & B \\ g_{31} & g_{32} & g_{33} & C \\ A & B & C & 0 \end{vmatrix} \right)^{1/2}},$$

and if the vector $\mathbf{n} = [A, B, C]$ is the unit vector, then

$$d = |A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)|.$$

In rectangular Cartesian coordinates,

$$d = \frac{|A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)|}{\sqrt{A^2 + B^2 + C^2}},$$

and if the vector $\mathbf{n} = \{A, B, C\} = [A, B, C]$ is the unit vector, then

$$d = |A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)|.$$

The cosines of the angles $\varphi_{1,2}$ between two planes specified by the equations

$$\mathbf{n}_1(\mathbf{r} - \mathbf{r}_1) = 0, \qquad \mathbf{n}_2(\mathbf{r} - \mathbf{r}_2) = 0$$

are computed from the formula

$$\cos \varphi_{1,2} = \pm \frac{|\mathbf{n}_1 \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|}.$$

If the planes are given by the equations

$$Ax + By + Cz + D = 0,$$
$$A'x + B'y + C'z + D' = 0$$

with respect to a rectangular Cartesian coordinate system, then

$$\cos \varphi_{1,2} = \pm \frac{AA' + BB' + CC'}{\sqrt{A^2 + B^2 + C^2}\sqrt{A'^2 + B'^2 + C'^2}}.$$

In general Cartesian coordinates we have

$$\cos \varphi_{1,2} = \pm \frac{([A, B, C],\ [A', B', C'])}{|[A, B, C]|\,|[A', B', C']|};$$

$$([A, B, C],\ [A', B', C']) = g^{11}AA' + g^{22}BB' + g^{33}CC'$$
$$+ g^{12}(AB' + A'B) + g^{23}(BC' + B'C) + g^{31}(CA' + C'A)$$
$$= -\frac{1}{g}\begin{vmatrix} g_{11} & g_{12} & g_{13} & A \\ g_{21} & g_{22} & g_{23} & B \\ g_{31} & g_{32} & g_{33} & C \\ A' & B' & C' & 0 \end{vmatrix},$$

$$|[A,\ B,\ C]| = \sqrt{g^{11}A^2 + g^{22}B^2 + g^{33}C^2 + 2g^{12}AB + 2g^{23}BC + 2g^{31}CA}$$
$$= \frac{1}{\sqrt{g}}\left(-\begin{vmatrix} g_{11} & g_{12} & g_{13} & A \\ g_{21} & g_{22} & g_{23} & B \\ g_{31} & g_{32} & g_{33} & C \\ A & B & C & 0 \end{vmatrix}\right)^{1/2},$$

$$|[A', B', C']|$$
$$= \sqrt{g^{11}A'^2 + g^{22}B'^2 + g^{33}C'^2 + 2g^{12}A'B' + 2g^{23}B'C' + 2g^{31}C'A'}$$
$$= \frac{1}{\sqrt{g}}\left(-\begin{vmatrix} g_{11} & g_{12} & g_{13} & A' \\ g_{21} & g_{22} & g_{23} & B' \\ g_{31} & g_{32} & g_{33} & C' \\ A' & B' & C' & 0 \end{vmatrix}\right)^{1/2}.$$

The parametric equation of a straight line passing through point $M_1(\mathbf{r}_1)$ and having the direction vector $\mathbf{a}$ is of the form

$$\mathbf{r} = \mathbf{r}_1 + t\mathbf{a},$$

where t is the coordinate of the point $M(\mathbf{r})$ on that line if we take point $M_1(\mathbf{r}_1)$ for the origin and vector $\mathbf{a}$ for the basis vector:

$$t = \frac{\overrightarrow{M_1M}}{\mathbf{a}}.$$

In a general Cartesian coordinate system, the parametric equations of a straight line are written as follows:

$$x = x_1 + tl,$$
$$y = y_1 + tm,$$
$$z = z_1 + tn,$$

where $M_1 = (x_1, y_1, z_1)$, $\mathbf{a} = \{l, m, n\}$.

The parametric equation of the straight line passing through two points $M_1(\mathbf{r}_1)$ and $M_2(\mathbf{r}_2)$ is of the form

$$\mathbf{r} = \mathbf{r}_1 + t(\mathbf{r}_2 - \mathbf{r}_1),$$

and, in general Cartesian coordinates,

$$x = x_1 + t(x_2 - x_1),$$
$$y = y_1 + t(y_2 - y_1),$$
$$z = z_1 + t(z_2 - z_1).$$

Two straight lines are said to be *coplanar* if they lie in one plane.

For two straight lines

$$\mathbf{r} = \mathbf{r}_1 + t\mathbf{a}, \qquad \mathbf{r}_2 = \mathbf{r}_2 + t\mathbf{b}$$

to be coplanar, it is necessary and sufficient that the following equality hold:

$$(\mathbf{r}_2 - \mathbf{r}_1, \mathbf{a}, \mathbf{b}) = 0$$

or, in general Cartesian coordinates,

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0,$$

where $M_1 = (x_1, y_1, z_1)$, $M_2 = (x_2, y_2, z_2)$, $\mathbf{a} = \{l_1, m_1, n_1\}$, $\mathbf{b} = \{l_2, m_2, n_2\}$.

Remark. The equations of a straight line are often written in the form

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

(canonical equations of a straight line). If one of the denominators is zero, then this notation is to be understood as follows: the numerator is also equal to zero. For example, the system of equations

$$\frac{x-2}{3}=\frac{y-5}{0}=\frac{z-1}{4}$$

is to be understood as follows:

$$y-5=0,$$
$$\frac{x-2}{3}=\frac{z-1}{4}.$$

The distance d from the point $M_0(\mathbf{r}_0)$ to the straight line given by the equation $\mathbf{r}=\mathbf{r}_1+t\mathbf{a}$ is computed from the formula

$$d=\frac{|(\mathbf{r}_1-\mathbf{r}_0,\mathbf{a})|}{|\mathbf{a}|}.$$

In rectangular Cartesian coordinates,

$$d=\frac{\sqrt{\Delta_1^2+\Delta_2^2+\Delta_3^2}}{\sqrt{l^2+m^2+n^2}},$$

where

$$\Delta_1=\begin{vmatrix} y_1-y_0 & z_1-z_0 \\ m & n \end{vmatrix},\quad \Delta_2=\begin{vmatrix} z_1-z_0 & x_1-x_0 \\ n & l \end{vmatrix},\quad \Delta_3=\begin{vmatrix} x_1-x_0 & y_1-y_0 \\ l & m \end{vmatrix},$$

and, in general Cartesian coordinates,

$$d=\sqrt{g}\,\frac{\sqrt{g^{11}\Delta_1^2+g^{22}\Delta_2^2+g^{33}\Delta_3^2+2g^{12}\Delta_1\Delta_2+2g^{23}\Delta_2\Delta_3+2g^{31}\Delta_3\Delta_1}}{\sqrt{g_{11}l^2+g_{22}m^2+g_{33}n^2+2g_{12}lm+2g_{23}mn+2g_{31}nl}}$$

$$=\frac{\left(-\begin{vmatrix} g_{11} & g_{12} & g_{13} & \Delta_1 \\ g_{21} & g_{22} & g_{23} & \Delta_2 \\ g_{31} & g_{32} & g_{33} & \Delta_3 \\ \Delta_1 & \Delta_2 & \Delta_3 & 0 \end{vmatrix}\right)^{1/2}}{\sqrt{g_{11}l^2+g_{22}m^2+g_{33}n^2+2g_{12}lm+2g_{23}mn+2g_{31}nl}}.$$

The shortest distance d between the noncollinear straight lines

$$\mathbf{r}=\mathbf{r}_1+t\mathbf{a},\qquad \mathbf{r}=\mathbf{r}_2+t\mathbf{b}$$

is

$$d=\frac{|(\mathbf{r}_2-\mathbf{r}_1,\mathbf{a},\mathbf{b})|}{|[\mathbf{ab}]|}.$$

If in rectangular Cartesian coordinates we have

$$\mathbf{r}_1 = \{x_1, y_1, z_1\},\ \mathbf{r}_2 = \{x_2, y_2, z_3\},\ \mathbf{a} = \{l_1, m_1, n_1\},\ \mathbf{b} = \{l_2, m_2, n_2\},$$

then

$$d = \frac{\operatorname{mod} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{\delta_1^2 + \delta_2^2 + \delta_3^2}},$$

where

$$\delta_1 = \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix},\quad \delta_2 = \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix},\quad \delta_3 = \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix},$$

and, in general Cartesian coordinates,

$$d = \frac{\operatorname{mod} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{g}\sqrt{g^{11}\delta_1^2 + g^{22}\delta_2^2 + g^{33}\delta_3^2 + 2g^{12}\delta_1\delta_2 + 2g^{23}\delta_2\delta_3 + 2g^{31}\delta_3\delta_1}}$$

$$= \frac{\operatorname{mod} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\left(- \begin{vmatrix} g_{11} & g_{12} & g_{13} & \delta_1 \\ g_{21} & g_{22} & g_{23} & \delta_2 \\ g_{31} & g_{32} & g_{33} & \delta_3 \\ \delta_1 & \delta_2 & \delta_3 & 0 \end{vmatrix}\right)^{1/2}}.$$

A straight line in space can be specified by the equations of two intersecting planes:

$$Ax + By + Cz + D = 0,$$
$$A'x + B'y + C'z + D' = 0$$

(general Cartesian coordinates).
Its direction vector is

$$\mathbf{a} = \left\{ \begin{vmatrix} B & C \\ B' & C' \end{vmatrix},\ \begin{vmatrix} C & A \\ C' & A' \end{vmatrix},\ \begin{vmatrix} A & B \\ A' & B' \end{vmatrix} \right\}.$$

To reduce the equations of the straight line

$$Ax + By + Cz + D = 0,$$
$$A'x + B'y + C'z + D' = 0$$

to parametric form, we have to find some solution x_1, y_1, z_1 of the system; and then the parametric equations of the given straight line are

$$x = x_1 + t\begin{vmatrix} B & C \\ B' & C' \end{vmatrix},$$

$$y = y_1 + t\begin{vmatrix} C & A \\ C' & A' \end{vmatrix},$$

$$z = z_1 + t\begin{vmatrix} A & B \\ A' & B' \end{vmatrix}$$

and the canonical equations are

$$\frac{x - x_1}{\begin{vmatrix} B & C \\ B' & C' \end{vmatrix}} = \frac{y - y_1}{\begin{vmatrix} C & A \\ C' & A' \end{vmatrix}} = \frac{z - z_1}{\begin{vmatrix} A & B \\ A' & B' \end{vmatrix}}.$$

A necessary and sufficient condition for the plane $Ax + By + Cz + D = 0$ and the straight line $x = x_1 + lt$, $y = y_1 + mt$, $z = z_1 + nt$, given by equations with respect to general Cartesian coordinates, to intersect, be parallel, or for the line to lie in the plane is as follows:

	condition
intersection	$Al + Bm + Cn \neq 0$
parallelism	$Al + Bm + Cn = 0$ $Ax_1 + By_1 + Cz_1 + D \neq 0$
straight line lies in plane	$Al + Bm + Cn = 0$ $Ax_1 + By_1 + Cz_1 + D = 0$

The angle between the straight line

$$\mathbf{r} = \mathbf{r}_1 + t\mathbf{a}$$

and the plane

$$\mathbf{n}(\mathbf{r} - \mathbf{r}_0) = 0$$

is given by the formula

$$\sin\varphi = \frac{|\mathbf{an}|}{|\mathbf{a}|\,|\mathbf{n}|}.$$

If, in rectangular Cartesian coordinates, a straight line is given by the equations

$$x = x_1 + lt,$$

$$y = y_1 + mt,$$

$$z = z_1 + nt$$

and a plane by the equation

$$Ax + By + Cz + D = 0,$$

then

$$\sin\varphi = \frac{Al + Bm + Cn}{\sqrt{A^2 + B^2 + C^2}\sqrt{l^2 + m^2 + n^2}}.$$

In general Cartesian coordinates we have

$$\sin\varphi = \frac{Al + Bm + Cn}{\sqrt{T_1}\sqrt{T_2}},$$

where

$$T_1 = g^{11}A^2 + g^{22}B^2 + g^{33}C^2 + 2g^{12}AB + 2g^{23}BC + 2g^{31}CA$$

$$= -\frac{1}{g}\begin{vmatrix} g_{11} & g_{12} & g_{13} & A \\ g_{21} & g_{22} & g_{23} & B \\ g_{31} & g_{32} & g_{33} & C \\ A & B & C & 0 \end{vmatrix},$$

$$T_2 = g_{11}l^2 + g_{22}m^2 + g_{33}n^2 + 2g_{12}lm + 2g_{23}mn + 2g_{31}nl.$$

A necessary and sufficient condition for a line and a plane to be parallel is of the form

$$[A, B, C] \parallel \{l, m, n\}$$

or

$$\{Ag^{11} + Bg^{12} + Cg^{13},\ Ag^{21} + Bg^{22} + Cg^{23},\ Ag^{31} + Bg^{32} + Cg^{33}\} \parallel \{l, m, n\}$$

or

$$\begin{aligned} Ag^{11} + Bg^{12} + Cg^{13} &= kl, \\ Ag^{21} + Bg^{22} + Cg^{23} &= km, \quad k \neq 0, \\ Ag^{31} + Bg^{32} + Cg^{33} &= kn, \end{aligned}$$

or

$$\begin{aligned} g_{11}l + g_{12}m + g_{13}n &= kA, \\ g_{21}l + g_{22}m + g_{23}n &= kB, \quad k \neq 0, \\ g_{31}l + g_{32}m + g_{33}n &= kC, \end{aligned}$$

and, in rectangular Cartesian coordinates,

$$A = kl, \quad B = km, \quad C = kn.$$

The *equation of a circle* (C, r) with centre $C(a, b)$ and radius r in rectangular Cartesian coordinates is of the form

$$(x - a)^2 + (y - b)^2 - r^2 = 0,$$

and if the centre of the circle is the origin of coordinates, then

$$x^2 + y^2 + z^2 = 0.$$

Sometimes the point $C(a, b)$ is regarded as a circle of zero radius (the *zero circle*). The equation of a zero circle $C(a, b)$ in rectangular Cartesian coordinates is of the form

$$(x - a)^2 + (y - b)^2 = 0.$$

For the coordinates x, y of any point (x, y) lying outside the circle

$$(x - a)^2 + (y - b)^2 - r^2 = 0$$

we have the inequality

$$\sigma = (x - a)^2 + (y - b)^2 - r^2 > 0,$$

and for all points $M(x, y)$ lying inside that circle we have

$$\sigma = (x - a)^2 + (y - b)^2 - r^2 < 0.$$

The number σ is called the *power of the point $M(x, y)$ with respect to the circle* (C, r) and is equal to

$$\sigma = d^2 - r^2,$$

where d is the distance from point M to the centre C of the circle (C, r). If an arbitrary straight line intersecting the circle (C, r) in two points A and B is drawn through M, then

$$\sigma = (\overrightarrow{MA}, \overrightarrow{MB}) = \overline{MA} \cdot \overline{MB}.$$

The *equation of a sphere* (S, r) with centre $S(a, b, c)$ and radius r in rectangular Cartesian coordinates is of the form

$$(x - a)^2 + (y - b)^2 + (z - c)^2 - r^2 = 0,$$

and if the centre of the sphere lies in the coordinate origin, then

$$x^2 + y^2 + z^2 - r^2 = 0.$$

The equations of zero spheres are:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = 0,$$

$$x^2 + y^2 + z^2 = 0$$

For the coordinates x, y, z of any point M lying outside the sphere

$$(x - a)^2 + (y - b)^2 + (z - c)^2 - r^2 = 0$$

we have the inequality

$$\sigma = (x - a)^2 + (y - b)^2 + (z - c)^2 - r^2 > 0,$$

and for all points $M(x, y, z)$ lying inside the sphere we have

$$\sigma = (x - a)^2 + (y - b)^2 + (z - c)^2 - r^2 < 0.$$

The number σ is called the *power of the point M with respect to the sphere* (S, r) and is equal to

$$\sigma = d^2 - r^2 = (\overrightarrow{MA}, \overrightarrow{MB}) = \overline{MA} \cdot \overline{MB},$$

where d is the distance between the points S and M, and A and B are points in which an arbitrary straight line passing through point M intersects the sphere (S, r).

Sec. 4. Complex numbers

In the set of complex numbers $x + yi$ (x and y assume all real values), the sum and product are defined as follows:

$$(x + yi) + (x' + y'i) = (x + x') + (y + y')\, i,$$

$$(x + yi)(x' + y'i) = (xx' - yy') + (xy' + x'y)\, i;$$

here, the real number x may be written as $x + 0 \cdot y$; in particular, $1 = 1 + 0i$, $0 = 0 + 0 \cdot i$. It is obvious that

$$0 + z = z, \quad 1 \cdot z = 1 \quad \text{and} \quad 0 \cdot z = 0$$

for all complex numbers.

The set of complex numbers contains all the real numbers $(x + 0 \cdot i = x + 0 = x)$ and also the number $i\,(0 + 1 \cdot i = 1 \cdot i = i)$, the square of which, by virtue of the definition of a product of complex numbers, is equal to -1:

$$i^2 = -1.$$

The operations of subtraction and division are defined as the inverses of addition and multiplication; if $z = x + yi$, then $-z = (-x) + (-y)\, i$. The following properties are obvious:

$$z + (z' + z'') = (z + z') + z'',$$

$$z + (-z) = 0,$$

$$z + z' = z' + z,$$

$$z(z'z'') = (zz')\, z'',$$

$$zz' = z'z$$

$$z(z' + z'') = zz' + zz'',$$

where z, z', z'' are arbitrary complex numbers.

The *modulus* (absolute value) $|z| = \rho$ of the complex number $z = x + yi$ is the principal root (positive real root) $\sqrt{x^2 + y^2}$:

$$|z| = \rho = \sqrt{x^2 + y^2}.$$

The *argument* (or amplitude) $\varphi = \arg z$ of the complex number $z = x + yi \neq 0$ is the number φ defined by the relations

$$\cos\varphi = \frac{x}{\rho} = \frac{x}{\sqrt{x^2+y^2}}, \quad \sin\varphi = \frac{y}{\rho} = \frac{y}{\sqrt{x^2+y^2}}.$$

The argument of the number $z \neq 0$ has an infinity of values. If φ is one of the values of the argument of $z \neq 0$, then all the values $\arg z$ are contained in the formula

$$\arg z \equiv \varphi + 2k\pi,$$

where k takes on all integer values.

This relation is frequently written as follows:

$$\arg z \equiv \varphi (\mathrm{mod}\ 2\pi)$$

(read: "the argument z is congruent to φ modulo 2π").

If $z = 0$, then $\rho = 0$, and φ is any number.

For two complex numbers $z \neq 0$ and $z' \neq 0$ to be the same, it is necessary and sufficient for their moduli to be equal,

$$|z| = |z'|,$$

and for their arguments to be congruent modulo 2π:

$$\arg z \equiv \arg z' (\mathrm{mod}\ 2\pi).$$

From the preceding formulas it follows that

$$z = x + yi = \rho(\cos\varphi + i\sin\varphi)$$

is the *trigonometric form of a complex number.*

The following formulas are valid: if

$$z = \rho(\cos\varphi + i\sin\varphi), \quad z' = \rho'(\cos\varphi' + i\sin\varphi'),$$

then

$$zz' = \rho\rho'[\cos(\varphi+\varphi') + i\sin(\varphi+\varphi')],$$

$$\frac{z}{z'} = \frac{\rho}{\rho'}[\cos(\varphi-\varphi') + i\sin(\varphi-\varphi')],\ z' \neq 0,$$

$$|zz'| = |z|\,|z'|,$$

$$\arg(zz') \equiv \arg z + \arg z' (\mathrm{mod}\ 2\pi),$$

$$\left|\frac{z}{z'}\right| = \frac{|z|}{|z'|},$$

$$\arg\frac{z}{z'} \equiv \arg z - \arg z' (\mathrm{mod}\ 2\pi), \quad z' \neq 0,$$

$$z^n = \rho^n(\cos n\varphi + i\sin n\varphi)$$

(n an integer); in particular, if $|z| = \rho = 1$, then

$$(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi) \text{ (the \textit{de Moivre formula})}.$$

If $z \neq 0$ and n is a natural number, then

$$\sqrt[n]{z} = \sqrt[n]{\rho(\cos\varphi + i\sin\varphi)} = \sqrt[n]{\rho}\left(\cos\frac{\varphi + 2k}{n} + i\sin\frac{\varphi + 2k\pi}{n}\right),$$

$$k = 0, 1, 2, \ldots, n - 1.$$

Two complex numbers $z = x + yi$ and $\bar{z} = x - yi$ are said to be *conjugate*. They have the following properties of conjugacy:

$$\bar{\bar{z}} = z,$$

$$\overline{z + z'} = \bar{z} + \bar{z}',$$

$$\overline{z - z'} = \bar{z} - \bar{z}',$$

$$\overline{zz'} = \bar{z}\bar{z}'.$$

If $u = \dfrac{z}{z'}$ $(z' \neq 0)$, then $\bar{u} = \dfrac{\bar{z}}{\bar{z}'}$

$$|z| = |\bar{z}|,$$

$$\arg\bar{z} \equiv -\arg z \pmod{2\pi}.$$

Suppose a rectangular Cartesian coordinate system is introduced in a plane. With every complex number $z = x + yi$ we associate a point $M(x, y)$. This correspondence is one-to-one.

The number z is called the *affix* of the point M. A point with affix z is here symbolized thus: $M(z)$ or $M = (z)$.

If $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$, $z_3 = x_3 + y_3 i$ are the affixes of the points A, B, C in rectangular Cartesian coordinates, then the area (ABC) of an oriented $\triangle \overrightarrow{ABC}$ is computed from the formula

$$(ABC) = \frac{i}{4}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}.$$

In particular, a necessary and sufficient condition for the collinearity of three points $A(z_1)$, $B(z_2)$, $C(z_3)$ is of the form

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0.$$

The transformation under which a point $M(z)$ is associated with a point $M'(z')$, where

$$z' = az + b,$$

is a *similarity transformation of the first kind* (that is, a transformation that does not change the orientation of the plane). Indeed,

$$z' = a\left(z + \frac{b}{a}\right)$$

and transition from point $M(z)$ to point $M'(z')$ is performed as a translation $z_1 = z + \dfrac{b}{a}$ and a transformation $z' = az_1$, which consists in a rotation about the origin through an angle $\arg a$ and a homothetic transformation with centre O and ratio $|a|$. All these instances are similarity transformations of the first kind.

A transformation under which a point $M(z)$ is associated with a point $M'(z')$, where $z' = a\bar{z} + b$, is a *similarity transformation of the second kind* (that is, a transformation that reverses the orientation). Indeed, this transformation consists in a symmetry $z_1 = \bar{z}$ with respect to the x-axis and a transformation $z' = az_1 + b$ that reduces to a translation, a rotation, and a homothetic transformation. Of all these transformations, only symmetry with respect to the x-axis changes the orientation of the plane.

From the foregoing it follows that if two triangles $\overrightarrow{ABC}$ and $\overrightarrow{PQR}$ are given via the affixes of their vertices,

$$A = (z_1), \quad B = (z_2), \quad C = (z_3),$$
$$P = (u_1), \quad Q = (u_2), \quad R = (u_3),$$

then a necessary and sufficient condition that $\triangle\overrightarrow{ABC}$ and $\triangle\overrightarrow{PQR}$ be similar and have the same orientation is

$$\begin{vmatrix} z_1 & u_1 & 1 \\ z_2 & u_2 & 1 \\ z_3 & u_3 & 1 \end{vmatrix} = 0,$$

and a necessary and sufficient condition that $\triangle\overrightarrow{ABC}$ and $\triangle\overrightarrow{PQR}$ be similar but have opposite orientations is

$$\begin{vmatrix} \bar{z}_1 & u_1 & 1 \\ \bar{z}_2 & u_2 & 1 \\ \bar{z}_3 & u_3 & 1 \end{vmatrix} = 0.$$

Let us consider two distinct points $M_1(z_1)$ and $M_2(z_2)$. On the basis of the foregoing, point $M(z)$ lies on the straight line M_1M_2 if and only if

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0.$$

This equation can therefore be called the equation of the straight line M_1M_2. It can be transformed to

$$z - z_1 = \frac{z_2 - z_1}{\bar{z}_2 - \bar{z}_1} (\bar{z} - \bar{z}_1).$$

We will call the ratio

$$\varkappa = \frac{z_2 - z_1}{\bar{z}_2 - \bar{z}_1}$$

the *complex slope of the straight line* M_1M_2. Note that

$$|\varkappa| = \left| \frac{z_2 - z_1}{\bar{z}_2 - \bar{z}_1} \right| = 1.$$

Thus, the equation of the straight line M_1M_2 can be written down as

$$z - z_1 = \varkappa(\bar{z} - \bar{z}_1),$$

where $|\varkappa| = 1$.

Conversely, any equation of the form

$$z - z_1 = \varkappa(\bar{z} - \bar{z}_1),$$

where $|\varkappa| = 1$, is an equation of a straight line. Indeed, since $|\varkappa| = 1$, it follows that

$$\varkappa = \cos \varphi + i \sin \varphi.$$

Setting up the equation of the straight line passing through two points with affixes z_1 and $z_1 + \cos \frac{\varphi}{2} + i \sin \frac{\varphi}{2}$, we obtain the equation

$$z - z_1 = \varkappa(\bar{z} - \bar{z}_1).$$

In particular, note the equation of the straight line that passes through the coordinate origin:

$$z = \varkappa \bar{z},$$

where

$$\varkappa = \cos \varphi + i \sin \varphi = \frac{z_1}{\bar{z}_1}$$

and $z_1 \neq 0$ is the affix of any point (z_1) of the straight line under consideration.

Note that the straight line

$$z = \varkappa \bar{z}$$

passes through points of the unit circle (the *unit circle* is a circle with centre at the coordinate origin and with radius equal to 1) with affixes $\sqrt{\varkappa}$ ($\sqrt{\varkappa}$ always has two values: they are the affixes of the endpoints of the diameter of the unit circle). Indeed, if $\sqrt{\varkappa}$ is either of the values of this radical, then for $z = \sqrt{\varkappa}$ the equation $z = \varkappa\bar{z}$ becomes an equality (the left-hand side is $\sqrt{\varkappa}$; the right-hand side is, $\varkappa\sqrt{\bar{\varkappa}} = \sqrt{\varkappa}$).

The two straight lines

$$z - z_1 = \varkappa(\bar{z} - \bar{z}_1), \qquad |\varkappa| = 1,$$

$$z - z_2 = \varkappa'(\bar{z} - \bar{z}_2), \qquad |\varkappa'| = 1,$$

are collinear if and only if $\varkappa = \varkappa'$. True enough, these lines are collinear if and only if the system of equations

$$z - \varkappa\bar{z} = z_1 - \varkappa\bar{z}_1,$$

$$z - \varkappa'\bar{z} = z_2 - \varkappa\bar{z}_2$$

in z, $\bar{z}$ either has no solution or has an infinitude of solutions, and this occurs if and only if

$$\begin{vmatrix} 1 & -\varkappa \\ 1 & -\varkappa' \end{vmatrix} = 0 \quad \text{or} \quad \varkappa = \varkappa'.$$

The two straight lines p and q given by the equations

$$z - z_1 = \varkappa(\bar{z} - \bar{z}_1), \qquad |\varkappa| = 1,$$

$$z - z_2 = \varkappa'(\bar{z} - \bar{z}_2), \qquad |\varkappa'| = 1,$$

are perpendicular if and only if

$$\varkappa + \varkappa' = 0.$$

Indeed, let us consider the straight lines p' and q', which are collinear respectively with p and q, but which pass through the coordinate origin:

$$p': \qquad z = \varkappa\bar{z},$$

$$q': \qquad z = \varkappa'\bar{z}$$

The straight lines p and q are perpendicular if and only if the lines p' and q' are perpendicular. Suppose p' and q' are perpendicular. On p' take a point with the affix $z_0 \neq 0$. Then by virtue of the relation

$$\arg(iz_0) \equiv \arg i + \arg z_0 \equiv \frac{\pi}{2} + \arg z_0 (\operatorname{mod} 2\pi)$$

the point with affix iz_0 lies on the straight line q'. We have

$$z_0 = \varkappa \bar{z}_0, \quad iz_0 = \varkappa' \overline{iz_0}$$

or

$$z_0 = \varkappa \bar{z}_0, \quad iz_0 = - \varkappa' i\bar{z}_0$$

or

$$z_0 = \varkappa \bar{z}_0, \quad z_0 = - \varkappa' \bar{z}_0.$$

Hence, $\varkappa \bar{z}_0 = - \varkappa' \bar{z}_0$ and since $z_0 \neq 0$, it follows that $\varkappa = - \varkappa'$, whence $\varkappa + \varkappa' = 0$.

Conversely, let $\varkappa + \varkappa' = 0$. Let us prove that the straight lines p' and q' are mutually perpendicular. Draw through the origin a straight line p^* perpendicular to q', and let $\varkappa^*$ be the complex slope of p^*. Then $\varkappa^* + \varkappa' = 0$, and since $\varkappa + \varkappa' = 0$, it follows that $\varkappa = \varkappa^*$ and, hence, the straight lines p^* and p' are coincident, that is $p' \perp q'$, whence $p \perp q$.

Remark. In analytic geometry, the *slope* k of a straight line that is noncollinear with the y-axis is the tangent of the angle of inclination of that line to the x-axis. If the complex slope of a straight line that is noncollinear with the y-axis is equal to

$$\varkappa = \cos \varphi + i \sin \varphi,$$

then the angle α of inclination of that line to the x-axis is equal to $\varphi/2$ (since a straight line passing through the coordinate origin and having a complex slope $\varkappa$ passes through points with affixes $\sqrt{\varkappa}$).

Now we find

$$\frac{1 - \varkappa}{1 + \varkappa} = - i \tan \frac{\varphi}{2} = - ik;$$

consequently,

$$k = i \cdot \frac{1 - \varkappa}{1 + \varkappa}.$$

Conversely,

$$\varkappa = \frac{i - k}{i + k}.$$

The angle formed by the straight line

$$z - z_1 = \varkappa(\bar{z} - \bar{z}_1)$$

and the straight line

$$z - z_2 = \varkappa'(\bar{z} - \bar{z}_2)$$

is equal to

$$\arg \frac{\sqrt{\varkappa'}}{\sqrt{\varkappa}} \pmod{\pi}.$$

Indeed, the straight lines $z = \varkappa\bar{z}$, $z = \varkappa'\bar{z}$, which are collinear with the given lines, pass through points with affixes $\sqrt{\varkappa}$ and $\sqrt{\varkappa'}$ respectively and so the angle formed by the two straight lines is equal to

$$\arg \sqrt{\varkappa'} - \arg \sqrt{\varkappa} \equiv \arg \frac{\sqrt{\varkappa'}}{\sqrt{\varkappa}} \pmod{\pi}.$$

The equation of any straight line can be written as

$$Az + B\bar{z} + C = 0,$$

where C is a real number and $B = \bar{A} \neq 0$. Conversely, any such equation is an equation of a straight line provided C is a real number and $B = \bar{A} \neq 0$

Proof. Let $Px + Qy + R = 0$ be the equation of a straight line in rectangular Cartesian coordinates. Since

$$x = \frac{1}{2}(z + \bar{z}),$$

$$y = \frac{1}{2i}(z - \bar{z}) = \frac{i}{2}(\bar{z} - z),$$

it can be rewritten thus:

$$\frac{1}{2}P(z + \bar{z}) + \frac{i}{2}Q(\bar{z} - z) + R = 0$$

or

$$\frac{P - Qi}{2}z + \frac{P + Qi}{2}\bar{z} + R = 0$$

or

$$Az + \bar{A}\bar{z} + C = 0 \qquad (C = 2R).$$

Conversely, setting

$$A = P - Qi,\ \bar{A} = B = P + Qi,$$

we can rewrite the equation

$$Az + B\bar{z} + C = 0$$

as

$$(P - Qi)(x + yi) + (P + Qi)(x - yi) + C = 0$$

or

$$2Px + 2Qy + C = 0,$$

which is a first-degree equation.

The equation

$$Az + \overline{A}\overline{z} + C = 0$$

is called a *self-conjugate equation of a straight line* since the left-hand side of the equation is a real function of x and y:

$$u = Az + \overline{A}\overline{z} + C,$$

$$\overline{u} = \overline{A}\overline{z} + Az + C = u.$$

If a straight line is given by the self-conjugate equation

$$Az + B\overline{z} + C = 0,$$

where $B = \overline{A} \neq 0$ and C is a real number, then the distance d from point (z_0) to this line is

$$d = \frac{|Az_0 + B\overline{z}_0 + C|}{2|A|}.$$

Indeed, the equation of the straight line passing through point (z_0) perpendicularly to the given straight line is of the form

$$z - z_0 = \frac{B}{A}(\overline{z} - \overline{z}_0)$$

or

$$Az - B\overline{z} - z_0A + \overline{z}_0B = 0.$$

From the system

$$Az + B\overline{z} + C = 0,$$

$$Az - B\overline{z} - Az_0 + B\overline{z}_0 = 0$$

we find the affix of the projection of point (z_0) on the given straight line:

$$z' = \frac{Az_0 - B\overline{z}_0 - C}{2A},$$

whence

$$z_0 - z' = \frac{Az_0 + B\overline{z}_0 + C}{2A}$$

and so

$$d = |z_0 - z'| = \frac{|Az_0 + B\overline{z}_0 + C|}{2|A|}.$$

If a straight line is given by the self-conjugate equation

$$Az + B\bar{z} + C = 0,$$

where $B = \bar{A} \neq 0$ and C is a real number, then for all points (z) lying to one side of that straight line we have

$$Az + B\bar{z} + C > 0$$

(positive half-plane), and for all points (z) lying to the other side of that line we have

$$Az + B\bar{z} + C < 0$$

(negative half-plane).

If (z_0) is any point lying on the straight line

$$Az + B\bar{z} + C = 0,$$

where $B = \bar{A} \neq 0$ and C is a real number, then the point $(z_0 + B)$ lies in the positive half-plane since

$$A(z_0 + B) + B(\bar{z}_0 + \bar{B}) + C = Az_0 + B\bar{z}_0 + C + AB + B\bar{B}$$

$$= AB + B\bar{B} = \bar{B}B + B\bar{B} = 2B\bar{B} = 2|B|^2 > 0.$$

If the sides of $\triangle ABC$ are given by the self-conjugate equations

$$(BC):\quad A_1 z + B_1\bar{z} + C_1 = 0,$$
$$(CA):\quad A_2 z + B_2\bar{z} + C_2 = 0,$$
$$(AB):\quad A_3 z + B_3\bar{z} + C_3 = 0,$$

where C_k are real numbers and $B_k = \bar{A}_k \neq 0$, then the area (ABC) of the oriented $\triangle \overrightarrow{ABC}$ is computed from the formula

$$(ABC) = \frac{i}{4}\,\frac{\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}^2}{\begin{vmatrix} A_2 & B_2 \\ A_3 & B_3 \end{vmatrix}\begin{vmatrix} A_3 & B_3 \\ A_1 & B_1 \end{vmatrix}\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}.$$

The result remains the same if the left-hand sides of the equations (BC), (CA), (AB) are multiplied by any complex numbers different from zero. The area S of $\triangle ABC$ is computed from the formula

$$S = \frac{1}{4}\cdot\frac{\operatorname{mod}\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}^2}{\operatorname{mod}\left(\begin{vmatrix} A_2 & B_2 \\ A_3 & B_3 \end{vmatrix}\begin{vmatrix} A_3 & B_3 \\ A_1 & B_1 \end{vmatrix}\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}\right)}.$$

LIST OF SYMBOLS

$\mathbf{a}$	vector
$\overrightarrow{AB}$	a vector with origin at point A and terminus at point B: a ray (radial line) with origin at A and passing through B
$\mathbf{0}$	zero vector
$\lvert\mathbf{a}\rvert = a$	magnitude (absolute value) of vector $\mathbf{a}$
$\lvert\overrightarrow{AB}\rvert = AB$	magnitude of vector $\overrightarrow{AB}$
$\mathbf{a} \parallel \mathbf{b}$	vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear
$\left.\begin{matrix}\mathbf{a}\uparrow\uparrow\mathbf{b}\\ \mathbf{a}\downarrow\downarrow\mathbf{b}\end{matrix}\right\}$	vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear and in the same direction
$\mathbf{a}\uparrow\downarrow\mathbf{b}$	vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear and in opposite directions
$\mathbf{a}_\varphi$	a vector obtained from vector $\mathbf{a}$ by a rotation through an angle φ in an oriented plane
$[\mathbf{a}]$	a vector obtained from vector $\mathbf{a}$ by a rotation through the angle $\pi/2$ in an oriented plane
$\left.\begin{matrix}(\mathbf{a},\ \mathbf{b})\\ \mathbf{a}\times\mathbf{b}\end{matrix}\right\}$	a pseudoscalar (or cross) product of vector $\mathbf{a}$ by vector $\mathbf{b}$ in an oriented plane
$(\mathbf{a}, \mathbf{b}, \mathbf{c})$	triple scalar product (or cross product or mixed product) of three vectors in oriented space
$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b}$	scalar product of vectors $\mathbf{a}$ and $\mathbf{b}$
$\mathbf{a}^2$	scalar square of vector $\mathbf{a}$
$[\mathbf{a}, \mathbf{b}]$	vector product of vector $\mathbf{a}$ by vector $\mathbf{b}$ in oriented space
$\mathbf{e}_1, \mathbf{e}_2$	general basis of vectors in a plane
$\mathbf{e}^1, \mathbf{e}^2$	dual basis of the basis $\mathbf{e}_1, \mathbf{e}_2$
$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	general basis of vectors in space
$\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$	dual basis of the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$
$\mathbf{i}, \mathbf{j}$	orthonormal basis of vectors in a plane
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	orthonormal basis of vectors in space
$\mathbf{a}^*, \mathbf{b}^*$	dual basis of the basis $\mathbf{a}, \mathbf{b}$ in a plane
$\mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*$	dual basis of the basis $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in space
$(O, \mathbf{e}_1, \mathbf{e}_2)$	general Cartesian system of coordinates in a plane
$(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$	general Cartesian system of coordinates in space
$(O, \mathbf{i}, \mathbf{j})$	rectangular Cartesian system of coordinates in a plane

$(O, \mathbf{i}, \mathbf{j}, \mathbf{k})$	rectangular Cartesian system of coordinates in space
$\{x, y\}$	vector in a plane specified by contravariant components (or coordinates) x, y
$\{x, y, z\}$	vector in space specified by contravariant components (or coordinates) x, y, z
$[x, y]$	vector in a plane specified by covariant components (or coordinates) x, y
$[x, y, z]$	vector in space specified by covariant components (or coordinates) x, y, z
(x, y)	point in a plane specified by coordinates x, y
(x, y, z)	point in space specified by coordinates x, y, z
AB	(1) a line segment, (2) a straight line passing through points A and B, (3) the magnitude of the vector $\overrightarrow{AB}$, (4) the length of the line segment AB
(AB)	oriented length of a line segment (length of segment AB on an oriented straight line with appended sign)
$\overrightarrow{ABC}$	oriented triangle
(ABC)	area of oriented $\triangle\, \overrightarrow{ABC}$ (area with appended sign)
$\overrightarrow{ABCD}$	oriented tetrahedron
$(ABCD)$	volume of oriented tetrahedron $\overrightarrow{ABCD}$ (volume with appended sign)
$\left.\begin{matrix}\overrightarrow{ABC} \uparrow\uparrow \overrightarrow{A'B'C'} \\ \overrightarrow{ABC} \downarrow\downarrow \overrightarrow{A'B'C'}\end{matrix}\right\}$	triangles with the same orientation
$\overrightarrow{ABC} \downarrow\uparrow \overrightarrow{A'B'C'}$	triangles with opposite orientations
(a, b)	(1) the straight line of intersection of planes a and b, (2) the oriented angle from straight line a to straight line b
$C(AB)D$; sometimes simply (AB)	dihedral angle with edge AB in the half-planes of which lie points C and D
$(ABCD) = \dfrac{\overrightarrow{AC}}{\overrightarrow{BC}} : \dfrac{\overrightarrow{AD}}{\overrightarrow{BD}}$	anharmonic (or cross) ratio of the points A, B, C, D
δ_i^j	Kronecker delta: $\delta_i^j = \begin{matrix} 0 \text{ for } i \neq j \\ 1 \text{ for } i = j \end{matrix}$
g_{ij}	fundamental tensor specified by covariant components

g^{ij}	fundamental tensor specified by contravariant components
g	Gram determinant
(O, r)	circle with centre at point O and radius r
(ABC)	circle passing through points A, B, C*
O, (O)	the centre of a circle circumscribed about a triangle, and the circle itself
I, (I)	the centre of a circle inscribed in a triangle with centre at point I, and the circle itself
I_a, I_b, I_c, (I_a), (I_b), (I_c)	the centres of circles inscribed in $\triangle ABC$, and the circles themselves
O_9, (O_9)	the centre of the nine-point circle (Euler's circle), and the circle itself
R	the radius of a circle (ABC)
r	the radius of a circle (I) inscribed in a triangle
r_a, r_b, r_c	the radii of circles (I_a), (I_b), (I_c) escribed in a triangle
$(ABCD)$	sphere passing through points A, B, C, D
$\sigma = \sigma(M, (O, r))$	the power of a point M with respect to a circle (O, r); $\sigma = MO^2 - r^2$
(O, k)	a homothetic transformation with centre at point O and ratio k
$[O, k]$	inversion with centre at point O and power of inversion k
$\lvert z \rvert$	modulus of a complex number z
$\arg z$	argument of a complex number z
$\bar{z}$	the conjugate complex of z
(z)	point having affix z
(z_0, R)	circle of radius R, the affix of the centre of which is z_0
$\equiv$	sign of equivalence
$\longleftrightarrow$	sign of a one-to-one mapping
$\rightleftarrows$	sign of a one-to-one correspondence
$\perp$	sign of perpendicularity
$\parallel$	sign of parallelism (collinearity)
$\#$	sign of equality and parallelism
$\uparrow\uparrow$, $\downarrow\downarrow$	signs of collinearity and identical direction
$\downarrow\uparrow$	sign of collinearity and opposite direction
σ_1, σ_2, σ_3	symmetric polynomials of affixes z_1, z_2, z_3
σ_1, σ_2, σ_3, σ_4	symmetric polynomials of affixes z_1, z_2, z_3, z_4
mod x	absolute value of x

* Thus, the notation (ABC) has the following distinct meanings: (1) the (oriented) area of $\triangle ABC$: (2) a circle passing through A, B, C (no confusion can result because the first meaning of the symbol is a number, the second is a figure).

Appendix

LIST OF BASIC FORMULAS FOR REFERENCES

Quadratic equations

1. $x^2 + px + q = 0;\ x_{1,2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$

$$ax^2 + bx + c = 0;\ x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (a \neq 0).$$

$$ax^2 + 2kx + c = 0;\ x_{1,2} = \frac{-k \pm \sqrt{k^2 - ac}}{a} \qquad (a \neq 0).$$

2. Vieta's formulas:

$$x_1 + x_2 = -p = -\frac{b}{a};\ x_1x_2 = q = \frac{c}{a}.$$

3. $x^2 + px + q = (x - x_1)(x - x_2);$

$ax^2 + bx + c = a(x - x_1)(x - x_2).$

Progressions

(a) *Arithmetic progression.*

1. General term of an arithmetic progression:

$$a_n = a_1 + (n - 1)d.$$

2. The sum of n terms of an arithmetic progression:

$$S_n = \left(\frac{a_1 + a_n}{2}\right) n = \left[\frac{2a_1 + (n - 1)d}{2}\right] n,$$

where d is the difference.

(b) *Geometric progression.*

1. General term of geometric progression:

$$u_n = u_1 q^{n-1}.$$

2. The sum of n terms of geometric progression:

$$S_n = \frac{u_1 - u_n q}{1 - q} = u_1 \frac{1 - q^n}{1 - q} = u_1 \frac{q^n - 1}{q - 1},$$

where q is the common ratio of the progression ($q \neq 1$).

3. The sum of an infinitely decreasing geometric progression:

$$S = \frac{u_1}{1-q}.$$

Logarithms

1. The notation $\log_a N = x$ is equivalent to the notation $ax = N$ $(a > 0,\ a \neq 1,\ N > 0)$ so that $a^{\log_a N} = N$.

2. $\log_a 1 = 0$. 3. $\log_a a = 1$. 4. $\log_a(N \cdot M) = \log_a N + \log_a M$. 5. $\log_a \frac{N}{M} = \log_a N - \log_a M$. 6. $\log_a N^n = n \log_a N (N > 0)$. 7. $\log_a \sqrt[n]{N} = \frac{1}{n} \log_a N$. 8. $\log_b N = \frac{\log_a N}{\log_a b}$.

Relationships between trigonometric functions of an angle

1. $\sin^2\alpha + \cos^2\alpha = 1$. 2. $\frac{\sin\alpha}{\cos\alpha} = \tan\alpha$. 3. $\frac{\cos\alpha}{\sin\alpha} = \cot\alpha$. 4. $\sin\alpha \cdot \operatorname{cosec}\alpha = 1$. 5. $\cos\alpha \cdot \sec\alpha = 1$. 6. $\tan\alpha \cdot \cot\alpha = 1$. 7. $1 + \tan^2\alpha = \sec^2\alpha$. 8. $1 + \cot^2\alpha = \operatorname{cosec}^2\alpha$.

Table of Signs and Selected Values of Trigonometric Functions

Function	Quadrants				I					II	III	IV
	I	II	III	IV	0°	30°	45°	60°	90°	180°	270°	360°
$\sin\alpha$	+	+	−	−	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	−1	0
$\cos\alpha$	+	−	−	+	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0	1
$\tan\alpha$	+	−	+	−	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	0	∞	0
$\cot\alpha$	+	−	+	−	∞	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	∞	0	∞

Table of Reduction Formulas

Angle / Function	$-\alpha$	$90° \mp \alpha$	$180° \mp \alpha$	$270° \mp \alpha$	$360° k \mp \alpha$
sin	$-\sin\alpha$	$+\cos\alpha$	$\pm\sin\alpha$	$-\cos\alpha$	$\mp\sin\alpha$
cos	$+\cos\alpha$	$\pm\sin\alpha$	$-\cos\alpha$	$\mp\sin\alpha$	$+\cos\alpha$
tan	$-\tan\alpha$	$\pm\cot\alpha$	$\mp\tan\alpha$	$\pm\cot\alpha$	$\mp\tan\alpha$
cot	$-\cot\alpha$	$\pm\tan\alpha$	$\mp\cot\alpha$	$\pm\tan\alpha$	$\mp\cot\alpha$

Transformations of trigonometric expressions

1. $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta;$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta;$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}.$$

2. $\sin 2\alpha = 2\sin\alpha\cos\alpha;$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1;$

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}.$$

3. $$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}};\quad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}};$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}.$$

4. $$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}};\quad \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}};\quad \tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}.$$

5. $$\cos^2\alpha = \frac{1+\cos 2\alpha}{2};\quad \sin^2\alpha = \frac{1-\cos 2\alpha}{2}.$$

6. $$\sin\alpha \pm \sin\beta = 2\sin\frac{\alpha\pm\beta}{2}\cos\frac{\alpha\mp\beta}{2};$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2};$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} = 2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2};$$

$$\tan\alpha \pm \tan\beta = \frac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta};\quad \cot\alpha \pm \cot\beta = \frac{\sin(\beta\pm\alpha)}{\sin\alpha\sin\beta}.$$

7. $$\cos mx\cdot\cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x];$$

$$\sin mx\cdot\sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x];$$

$$\sin mx\cdot\cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x].$$

Inverse trigonometric functions

1. $-\frac{\pi}{2} \leqslant \arcsin x \leqslant \frac{\pi}{2}, \ \sin(\arcsin x) = x.$

2. $0 \leqslant \arccos x \leqslant \pi, \ \cos(\arccos x) = x.$

3. $-\frac{\pi}{2} \arctan x < \frac{\pi}{2}, \ \tan(\arctan x) = x.$

4. $0 < \operatorname{arccot} x < \pi, \ \cot(\operatorname{arccot} x) = x.$

Elementary trigonometric equations

1. $\sin x = a, \ x = (-1)^n \arcsin a + \pi n.$
2. $\cos x = a, \ x = \pm \arccos a + 2\pi n.$
3. $\tan x = a, \ x = \arctan a + \pi n.$ $(n = 0, \pm 1, \pm 2 \dots)$
4. $\cot x = a, \ x = \operatorname{arccot} a + \pi n.$

Symmetric Polynomials

For three numbers z_1, z_2, z_3

$$\sigma_1 = z_1 + z_2 + z_3,$$
$$\sigma_2 = z_1 z_2 + z_2 z_3 + z_3 z_1,$$
$$\sigma_3 = z_1 z_2 z_3.$$

For four numbers z_1, z_2, z_3, z_4

$$\sigma_1 = z_1 + z_2 + z_3 + z_4,$$
$$\sigma_2 = z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4,$$
$$\sigma_3 = z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4,$$
$$\sigma_4 = z_1 z_2 z_3 z_4.$$

Kronecker delta

$$\delta_i^j = \begin{cases} 0, & \text{if} \quad i \neq j, \\ 1, & \text{if} \quad i = j. \end{cases}$$

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NAME INDEX

SUBJECT INDEX

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